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# Electromagnetic scattering by small dielectric inhomogeneities ☆

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**Abstract**

In this work we carefully derive accurate asymptotic expansions of the electric and magnetic fields, the resonant frequencies, and the scattering amplitude in the practically interesting situation, where a number of dielectric objects of small diameter and with different material characteristics are imbedded in an otherwise smooth medium.


**Résumé**

Nous obtenons des formules asymptotiques précises pour les perturbations des champs électrique et magnétique, des fréquences de résonance ainsi que de l'amplitude de diffusion qui sont dues à la présence de petites inhomogénéités diélectriques.


*Keywords:* Helmholtz equation; Maxwell's equations; Electromagnetic fields; Resonant frequencies; Scattering amplitude; Small dielectric inhomogeneities

*MSC:* 35B40

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## 1. Introduction

Let $\Omega$ be a bounded, open subset of $\mathbb{R}^3$, with a smooth boundary. If the domain $\Omega$ is occupied by a material of magnetic permeability $\mu$, and electric permittivity $\varepsilon$, then the homogeneous, time-dependent, linear Maxwell Equations take the form:

$$\operatorname{curl} \mathbf{E} = -\mu \frac{\partial}{\partial t} \mathbf{H} \quad \text{in } \Omega, \qquad \operatorname{curl} \mathbf{H} = \varepsilon \frac{\partial}{\partial t} \mathbf{E} \quad \text{in } \Omega,$$

$\mathbf{E} \in \mathbb{R}^3$, and $\mathbf{H} \in \mathbb{R}^3$, is the electric field and the magnetic field respectively. In this work we shall only consider time-harmonic solutions to the above equations, i.e., special solutions of the form:

$$\mathbf{E}(x,t) = \operatorname{Re}\{E(x)\mathrm{e}^{-\mathrm{i}\omega t}\}, \quad \text{and} \quad \mathbf{H}(x,t) = \operatorname{Re}\{H(x)\mathrm{e}^{-\mathrm{i}\omega t}\}, \quad x \in \Omega,\ t > 0,$$

where $\omega > 0$ denotes the given frequency, and where the $\mathbb{C}^3$ valued fields $E(x)$ and $H(x)$ satisfy:

$$\operatorname{curl} E = \mathrm{i}\omega\mu H \quad \text{in } \Omega, \qquad \operatorname{curl} H = -\mathrm{i}\omega\varepsilon E \quad \text{in } \Omega.$$

We eliminate the magnetic field from the above equations by dividing the first equation by $\mu$ and taking the curl to obtain the following system of equations for $E$:

$$\operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl} E\right) - \omega^2 \varepsilon E = 0 \quad \text{in } \Omega. \tag{1.1}$$

In order to arrive at particular nontrivial solutions to these equations, we shall prescribe nontrivial boundary conditions for $E \times \nu$ on the boundary of the domain $\Omega$ ($\nu$ denotes the outward unit normal to $\Omega$). Having found the electric field $E$, we then obtain the magnetic field $H$ through the formula

$$H = \frac{1}{\mathrm{i}\omega\mu} \operatorname{curl} E.$$

In order to insure well-posedness of that boundary value problem we shall always assume that $\omega$ is not a *resonant frequency*. We define a resonant frequency to be such that there exists a nontrivial electric field $E$ that is solution to (1.1) with $E \times \nu = 0$ on $\partial\Omega$.

We will also discuss the *scattering problem* in the whole of $\mathbb{R}^3$ where the coefficients $\varepsilon$ and $\mu$ are assumed to take positive constant values $\varepsilon_e$ and $\mu_e$ outside the domain $\Omega$. To formulate this problem, we consider:

$$E_{in}(x) = \mathrm{i}k_e(q \times p) \times q\,\mathrm{e}^{\mathrm{i}k_e q\cdot x}$$

to be an incident plane wave, where $k_e = \omega\sqrt{\varepsilon_e\mu_e}$, $q \in \mathbb{R}^3$ is a unit vector giving the direction of propagation and $p \in \mathbb{R}^3$ is a constant vector giving the polarization. The scattering problem consists in finding solution $E$ of the Maxwell equations:

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) - \omega^2 \varepsilon E = 0 \tag{1.2}$$

in all of $\mathbb{R}^3$, subject to the radiation condition as $|x| \to +\infty$:

$$\left|\frac{\partial}{\partial |x|}(E - E_{in}) - \mathrm{i}k_e(E - E_{in})\right| = \mathrm{O}\left(\frac{1}{|x|^2}\right).$$

For the above scattering problem we will consider the *scattering amplitude*, $E_\infty(x/|x|, p, q, \omega)$, defined as the function which satisfies:

$$E(x) = E_{in}(x) + \frac{\mathrm{e}^{\mathrm{i}k_e|x|}}{|x|} E_\infty\left(\frac{x}{|x|}, p, q, \omega\right) + \mathrm{O}\left(\frac{1}{|x|^2}\right)$$

as $|x| \to +\infty$.

If we now focus on a special case of the three-dimensional Maxwell equations, namely the case where the coefficients $\varepsilon, \mu$ and the fields $E, H$ are independent of one of the variables, say $x_3$ (and the domain takes the form of a cylinder parallel to the $x_3$-axis). In this case the Maxwell equations split into two sets of independent equations, one for the fields $E^* = (0, 0, E_3)$, $H^* = (H_1, H_2, 0)$ and one for the fields $E^{**} = (E_1, E_2, 0)$, $H^{**} = (0, 0, H_3)$. The first set of equations are associated with the terminology *TE* (transverse electric), the second set with the terminology TM (transverse magnetic).

Consider the TE situation. Let $\widetilde{\Omega}$ denote the cross section of the (vertical) cylinder, and let $\tilde{x}$ denote the "cross sectional coordinates" $\tilde{x} = (x_1, x_2)$. The equation for $E^*$ transforms into the following scalar Helmholtz equation:

$$\operatorname{div}_{\tilde{x}}\left(\frac{1}{\mu(\tilde{x})}\operatorname{grad}_{\tilde{x}} E_3(\tilde{x})\right) + \omega^2 \varepsilon(\tilde{x}) E_3(\tilde{x}) = 0 \tag{1.3}$$

for $E_3$, with the corresponding magnetic field $H^*$ given by:

$$H^*(\tilde{x}) = \frac{1}{\mathrm{i}\omega\mu}\left(\frac{\partial E_3}{\partial x_2}, -\frac{\partial E_3}{\partial x_1}, 0\right).$$

Knowing $E^* \times \nu$ on the (vertical) boundary of the three-dimensional domain amounts to knowing $E_3$ on the boundary of $\widetilde{\Omega}$.

Consider now the TM situation. The equation for $H^{**}$ transforms into the following scalar equation

$$\operatorname{div}_{\tilde{x}}\left(\frac{1}{\varepsilon}\operatorname{grad}_{\tilde{x}} H_3\right) + \omega^2 \mu H_3 = 0 \tag{1.4}$$

for $H_3$, with the corresponding electric field $E^{**}$ given by:

$$E^{**} = \frac{\mathrm{i}}{\omega\varepsilon}\left(\frac{\partial H_3}{\partial x_2}, -\frac{\partial H_3}{\partial x_1}, 0\right).$$

Knowing $E^{**} \times \nu$ on the (vertical) boundary of the three-dimensional domain amounts to knowing $\frac{1}{\varepsilon}\partial H_3/\partial\nu$ on $\partial\widetilde{\Omega}$.

The *eigenvalue problem* that will be of interest in the TE case (respectively TM case) consists on finding the (resonant) frequencies $\omega$ such that there exists nontrivial solutions $E_3$ (respectively $H_3$) with $E_3 = 0$ (respectively $\frac{1}{\varepsilon}\partial H_3/\partial\nu = 0$) on $\partial\widetilde{\Omega}$.

We will also investigate the *scattering problem* (in the TE case) which may be formulated as follows: find $E_3(\tilde{x})$ solution to

$$\operatorname{div}_{\tilde{x}}\left(\frac{1}{\mu(\tilde{x})}\operatorname{grad}_{\tilde{x}} E_3(\tilde{x})\right) + \omega^2\varepsilon(\tilde{x})E_3(\tilde{x}) = 0 \tag{1.5}$$

in all of $\mathbb{R}^2$, subject to the radiation condition as $|\tilde{x}| \to +\infty$:

$$\left|\frac{\partial}{\partial|\tilde{x}|}\big(E_3(\tilde{x}) - \widetilde{E}_{in}(\tilde{x})\big) - \mathrm{i}k_e\big(E_3(\tilde{x}) - \widetilde{E}_{in}(\tilde{x})\big)\right| = \mathrm{O}\left(\frac{1}{|\tilde{x}|}\right),$$

where $\widetilde{E}_{in}(\tilde{x}) = \mathrm{e}^{\mathrm{i}k_e\tilde{q}\cdot\tilde{x}}$. Here $\tilde{q} \in \mathbb{R}^2$ is a unit vector giving the direction of propagation. For this problem, particular attention will be devoted to the *scattering amplitude*, $A(\tilde{x}/|\tilde{x}|, \tilde{q}, \omega)$, that is defined to be the function which satisfies:

$$E_3(\tilde{x}) = \widetilde{E}_{in}(\tilde{x}) + \frac{\mathrm{e}^{\mathrm{i}k_e|\tilde{x}|}}{|\tilde{x}|^{1/2}}A\left(\frac{\tilde{x}}{|\tilde{x}|}, \tilde{q}, \omega\right) + \mathrm{O}\left(\frac{1}{|\tilde{x}|^{3/2}}\right)$$

as $|\tilde{x}| \to +\infty$.

In this work we carefully derive *very accurate asymptotic expansions* of

(i) the electric and magnetic fields,
(ii) the resonant frequencies,
(iii) the scattering amplitude,

in the practically very interesting situation, where a number of dielectric objects of *small diameter* and with different material characteristics are imbedded in an otherwise smooth medium.

Suppose that $\omega$ is not a resonant frequency we begin our analysis in Section 2 by rigorously deriving *high-order asymptotic expansions* of the electric and magnetic fields which are *valid uniformly in space*. The leading-order term in these asymptotic expansions has been derived by Vogelius and Volkov [47] and Ammari et al. [8]; see also the prior works of Cedio-Fengya et al. [15] for the conductivity problem and Friedman and Vogelius [18] for the case of perfectly conducting or insulating inhomogeneities. The higher-order terms are essential when the leading-order term in the asymptotic expansion of the electromagnetic fields, given in [8,47], vanishes. Our analysis leading to high-order uniform asymptotic expansions is considerably different from those in [3,8,11,15, 18,27,29,31,47]. Our asymptotic formulas involve polarization tensors associated with the electromagnetic inhomogeneities that seem to be natural generalizations of the tensors that have been introduced by Schiffer and Szegö [44] and thoroughly studied by many other authors [15,18,26,32,33,42].

Combining our asymptotic formulas given in this section with the boundary perturbation method of Bruno and Reitich [13,14] will provide a very efficient tool for solving the electromagnetic scattering problem in complex domains.

In Section 3 we study in some detail the properties of these generalized polarization tensors. In particular, we rigorously establish some useful properties of these generalized polarization tensors and carefully study their limits corresponding to extreme conductivity cases where the dielectric inhomogeneities are either perfectly insulating or perfectly conducting.

Section 4 considers the effect of these small dielectric inhomogeneities on the resonant frequencies. We will extend the calculations of [6] to obtain further terms in the expansions of the weighted mean of the perturbed resonant frequencies. In a series of papers [36–41], Ozawa derived the leading-order term in the asymptotic expansions of simple eigenvalues in the special case of conductivity inhomogeneities that are either perfectly insulating balls or perfectly conducting balls. The nondegeneracy condition for the nonperturbed eigenvalue was essential in Ozawa's proofs. In Section 4, we allow inhomogeneities with more general shapes (star-shaped domains) and finite electromagnetic parameters, remove the condition that the eigenvalue is simple, and provide more accurate asymptotic expansions. To rigorously prove these accurate asymptotic formulas we combine the expansions derived in Section 2, additional technical estimates, and a result about the convergence of eigenvalues of a sequence of self-adjoint collectively compact operators.

The aim of Section 5 is to provide rigorous derivations of asymptotic formulas for the scattering amplitudes. Our technique for studying the scattering amplitudes is to reduce the scattering problem to a bounded domain with the aid of integral equation methods.

We conclude this work by formally extending in Section 6 some of the results obtained for the Helmholtz equation to the full Maxwell's equations.

The reader is referred to the recent papers [2,4] for rigorous derivations of asymptotic formulas for the displacement vector in an elastic medium consisting of finitely many imperfections of small diameter, imbedded in a homogeneous reference medium.

For simplicity we assume throughout this paper that the material characteristics of each of the inhomogeneities as well as those of the background media are constants. This simplification allows us to base our analysis largely on boundary integral methods; it also allows us to use explicit fundamental solutions for the underlying Helmholtz and Maxwell equations.

The results we provide for the Helmholtz equation hold for two and three dimensions, but with minimal changes they could be extended to higher dimensions as well.

Making use of the accurate asymptotic expansions derived in this paper we certainly would be able to identify the small inhomogeneities with high resolution from boundary or spectral informations and dramatically improve the reconstruction algorithm described in [7].

## 2. Asymptotic expansions of the solutions

### 2.1. Problem formulation

In the introduction we have briefly made reference to some of the ingredients of the electromagnetic wave propagation theory. In this section we shall provide more concise statements of the electromagnetic problems we are dealing with. Let $\Omega$ be a bounded $\mathcal{C}^\infty$-domain in $\mathbb{R}^d$, $d = 2$ or 3. The outward unit normal to $\partial\Omega$ is denoted by $\nu$. Assume that $\Omega$ contains a finite number of inhomogeneities, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbb{R}^d$ is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is:

$$\mathcal{B}_\alpha = \bigcup_{j=1}^{m} \{z_j + \alpha B_j\}.$$

The points $z_j \in \Omega$, $j = 1, \ldots, m$, which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

$$|z_j - z_l| \geqslant c_0 > 0, \quad \forall j \neq l, \quad \text{and} \quad \operatorname{dist}(z_j, \partial\Omega) \geqslant c_0 > 0, \quad \forall j. \tag{2.1}$$

Assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to $\mathbb{R}^d \setminus \overline{\Omega}$ is larger than $c_0/2$. Let $\mu_0$ and $\varepsilon_0$ denote the permeability and the permittivity of the background medium, and assume that $\mu_0 > 0$ *and* $\varepsilon_0 > 0$ *are positive constants*. Let $\mu_j > 0$ and $\varepsilon_j > 0$ denote the permeability and the permittivity of the $j$th inhomogeneity, $z_j + \alpha B_j$; these are also assumed to be positive constants. Introduce the piecewise-constant magnetic permeability

$$\mu_\alpha(x) = \begin{cases} \mu_0, & x \in \Omega \setminus \overline{\mathcal{B}_\alpha}, \\ \mu_j, & x \in z_j + \alpha B_j, \ j = 1, \ldots, m. \end{cases} \tag{2.2}$$

If we allow the degenerate case $\alpha = 0$, then the function $\mu_0(x)$ equals the constant $\mu_0$. The piecewise constant electric permittivity, $\varepsilon_\alpha(x)$ is defined analogously.

Consider solutions to the time-harmonic Maxwell's equations with TE symmetry and $\exp(-\mathrm{i}\omega t)$ time dependence. Let $E_\alpha$ be the electric field (or rather, the transversal strength) in the presence of the inhomogeneities. It satisfies the Helmholtz equation:

$$\operatorname{div}\left(\frac{1}{\mu_\alpha} \operatorname{grad} E_\alpha\right) + \omega^2 \varepsilon_\alpha E_\alpha = 0 \quad \text{in } \Omega, \tag{2.3}$$

with the boundary condition $E_\alpha = f$ on $\partial\Omega$, where $\omega > 0$ is a given frequency. The electric field, $E_0$, in the absence of any inhomogeneities, satisfies the following equation:

$$\Delta E_0 + k^2 E_0 = 0 \quad \text{in } \Omega, \tag{2.4}$$

where $k^2 = \omega^2 \mu_0 \varepsilon_0$, with $E_0 = f \in H^{1/2}(\partial\Omega)$ on $\partial\Omega$. In order to insure well-posedness (also for the $\alpha$-dependent case for $\alpha$ sufficiently small as it will be proven in Lemma 2.1) we shall assume that

$$\begin{aligned} &k^2 \text{ is not an eigenvalue for the operator } -\Delta \text{ in } L^2(\Omega) \\ &\quad\text{with homogeneous Dirichlet boundary conditions.} \end{aligned} \tag{2.5}$$

In order to define the natural weak formulation of the problem (2.3), let $a_\alpha^{\mathrm{TE}}$ denote the sesquilinear form:

$$a_\alpha^{\mathrm{TE}}(u, v) = \int_\Omega \frac{1}{\mu_\alpha} \operatorname{grad} u \cdot \operatorname{grad} v - \omega^2 \int_\Omega \varepsilon_\alpha u v, \tag{2.6}$$

defined on $H_0^1(\Omega) \times H_0^1(\Omega)$. Let $b$ be a given conjugate-linear functional on $H_0^1(\Omega)$. Our assumption (2.5) is that the variational problem:

$$a_0^{\mathrm{TE}}(u, v) = b(v) \quad \text{for all } v \in H_0^1(\Omega),$$

has a unique solution. The following lemma from [47] shows that the assumption (2.5) also leads to the unique solvability of (2.3).

**Lemma 2.1.** *Suppose* (2.1) *and* (2.5) *are satisfied, and let* $a_\alpha^{\mathrm{TE}}$, $0 \leqslant \alpha$, *be the sesquilinear forms introduced by* (2.6). *There exists a constant* $0 < \alpha_0$, *such that given any* $0 \leqslant \alpha < \alpha_0$, *and any bounded, conjugate-linear functional,* $b$, *on* $H_0^1(\Omega)$, *there is a unique* $u_\alpha \in H_0^1(\Omega)$ *which satisfies* $a_\alpha^{\mathrm{TE}}(u_\alpha, v) = b(v)$ *for all* $v \in H_0^1(\Omega)$. *Furthermore, there exists a constant* $C$, *independent of* $\alpha$ *and* $b$, *such that*

$$\|u_\alpha\|_{H^1(\Omega)} \leqslant C \sup_{v \in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)} = 1} \left| b(v) \right|.$$

**Proof.** In order to prove this lemma it is convenient to introduce a decomposition of $a_\alpha^{\mathrm{TE}}$. Pick a fixed positive constant, $\lambda$, with $\lambda > \omega^2$, and write $a_\alpha^{\mathrm{TE}}$ as $a_\alpha^{\mathrm{TE}} = A_\alpha + B_\alpha$, where

$$A_\alpha(u, v) = \int_\Omega \frac{1}{\mu_\alpha} \operatorname{grad} u \cdot \operatorname{grad} v + \left(\lambda - \omega^2\right) \int_\Omega \varepsilon_\alpha u v$$

and

$$B_\alpha(u, v) = -\lambda \int_\Omega \varepsilon_\alpha u v.$$

Suppose (2.5) satisfied, the sesquilinear form $A_\alpha$ is uniformly continuous and uniformly coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$. It is also convenient to introduce a family of bounded linear operators $K_\alpha : H_0^1(\Omega) \to H_0^1(\Omega)$ by:

$$A_\alpha(K_\alpha u, v) = B_\alpha(u, v) = -\lambda \int_\Omega \varepsilon_\alpha u v,$$

for all $u$ and $v$ in $H_0^1(\Omega)$.

Let $\alpha_n$ be a sequence converging to zero. We first show that the linear operators $\{K_{\alpha_n}\}$ are *collectively compact* and $K_{\alpha_n}$ *converges pointwise* to $K_0$ as $\alpha_n$ approaches 0.

We remind the reader that the operators $\{K_{\alpha_n}\}$ are collectively compact iff the set $\{K_{\alpha_n}(u)\colon\ n \geqslant 1,\ u \in H_0^1(\Omega),\ \|u\|_{H^1(\Omega)} \leqslant 1\}$ is relatively compact (i.e., its closure is compact) in $H_0^1(\Omega)$.

Fix $u \in H_0^1(\Omega)$, then

$$A_{\alpha_n}\big((K_{\alpha_n} - K_0)u, v\big) = B_{\alpha_n}(u, v) - B_0(u, v) + A_0(K_0 u, v) - A_{\alpha_n}(K_0 u, v),$$

for all $v \in H_0^1(\Omega)$. We easily see that

$$\sup_{v \in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)} = 1} \big|B_0(u, v) - B_{\alpha_n}(u, v)\big| \to 0, \tag{2.7}$$

as $\alpha_n \to 0$. It is also clear that

$$\sup_{v \in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)} = 1} \big|A_0(K_0 u, v) - A_{\alpha_n}(K_0 u, v)\big| \to 0, \tag{2.8}$$

as $\alpha_n \to 0$, and a combination of (2.7) and (2.8) yields

$$\sup_{v \in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)} = 1} \big|A_{\alpha_n}\big((K_{\alpha_n} - K_0)u, v\big)\big| \to 0,$$

as $\alpha_n \to 0$. Since $A_{\alpha_n}$ is uniformly coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$, it follows now that

$$\big\|(K_{\alpha_n} - K_0)u\big\|_{H^1(\Omega)} \to 0,$$

as $\alpha_n \to 0$. This verifies the pointwise convergence of the operators $\{K_{\alpha_n}\}$.

Let $K_{\tau_m}(u_m)$ be any sequence from the set

$$\big\{K_{\alpha_n}(u)\colon\ n \geqslant 1,\ u \in H_0^1(\Omega),\ \|u\|_{H^1(\Omega)} \leqslant 1\big\}.$$

In order to verify the collective compactness of the operators $\{K_{\alpha_n}\}$ we need to show that the sequence $K_{\tau_m}(u_m)$ contains a convergent subsequence. By extraction of a subsequence (still referred to as $K_{\tau_m}(u_m)$) we may assume that either:

(1) $\tau_m = \tau$ is constant (i.e., independent of $m$), or
(2) $\tau_m \to 0$ as $m \to +\infty$.

We may also assume that $u_m$ converges weakly to some $u_\infty \in H_0^1(\Omega)$. We introduce the sequence $u'_m = u_m - u_\infty$. Clearly $\|u'_m\|_{H^1(\Omega)} \leqslant 2$ and $u'_m$ converges weakly to zero. Since the imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, this gives that $u'_m$ has a strongly convergent subsequence to zero in $L^2(\Omega)$ (still referred to as $u'_m$). From the definition of $K_{\tau_m}$ it follows immediately that

$$\sup_{v\in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)}=1} \left|A_{\tau_m}\left(K_{\tau_m}u'_m, v\right)\right| = \sup_{v\in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)}=1} \left|B_{\tau_m}\left(u'_m, v\right)\right| \leqslant C\left\|u'_m\right\|_{L^2(\Omega)}.$$

Since $A_{\tau_m}$ is uniformly coercive and since $\|u'_m\|_{L^2(\Omega)} \to 0$ we conclude from the above estimate that $\|K_{\tau_m}u'_m\|_{H^1(\Omega)} \to 0$, which is exactly what we are aiming at.

We want now to solve the variational problem:

$$\text{find } u_\alpha \in H_0^1(\Omega) \text{ such that } a_\alpha^{\mathrm{TE}}(u_\alpha, v) = b(v) \quad \text{for all } v \in H_0^1(\Omega). \tag{2.9}$$

This problem can be rewritten as

$$A_\alpha(u_\alpha, v) + B_\alpha(u_\alpha, v) = b(v) \quad \text{for all } v \in H_0^1(\Omega),$$

or as

$$A_\alpha\big((I + K_\alpha)u_\alpha, v\big) = b(v) \quad \text{for all } v \in H_0^1(\Omega). \tag{2.10}$$

Since $A_\alpha$ is uniformly continuous and coercive on $H_0^1(\Omega)$ it now follows that the variational problem (2.10) is equivalent to find $u_\alpha \in H_0^1(\Omega)$ such that

$$(I + K_\alpha)u_\alpha = F_\alpha.$$

Here the function $F_\alpha \in H_0^1(\Omega)$ is defined by $A_\alpha(F_\alpha, v) = b(v)$ for all $v \in H_0^1(\Omega)$, and therefore satisfies:

$$\|F_\alpha\|_{H^1(\Omega)} \leqslant C \sup_{v\in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)}=1} \left|b(v)\right|. \tag{2.11}$$

By the same arguments as we just went through earlier in this proof, the original variational problem: find $u_0 \in H_0^1(\Omega)$ such that $a_0^{\mathrm{TE}}(u_0, v) = b(v)$ for all $v \in H_0^1(\Omega)$ is thus equivalent to find $u_0 \in H_0^1(\Omega)$ such that $(I + K_0)u_0 = F_0$ with $F_0 \in H_0^1(\Omega)$ defined by $A_0(F_0, v) = b(v)$ for all $v \in H_0^1(\Omega)$. The fact that this problem has a unique solution (assumption (2.5)) implies that $I + K_0$ is an invertible operator. For any sequence $\alpha_n$ converging to zero we have already verified that the operators $\{K_{\alpha_n}\}$ are collectively compact and converge pointwise to $K_0$. From the theory of collectively compact operators

[10] it now follows that there exists a constant $0 < \alpha_0$, such that given any $0 \leqslant \alpha < \alpha_0$, the operator $I + K_\alpha$ is invertible with

$$\left\|(I + K_\alpha)^{-1} F\right\|_{H^1(\Omega)} \leqslant C \|F_\alpha\|_{H^1(\Omega)} \tag{2.12}$$

for some constant $C$, independent of $\alpha$. Based on (2.11) and (2.12) it follows immediately that the variational problem (2.9) has a unique solution $u_\alpha \in H_0^1(\Omega)$ satisfying

$$\|u_\alpha\|_{H^1(\Omega)} \leqslant C \sup_{v \in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)}=1} \left|b(v)\right|.$$

Thus the proof of Lemma 2.1 is complete. □

Using the above lemma we can show as in [47] that the following holds.

**Proposition 2.1.** *Suppose* (2.1) *and* (2.5) *are satisfied. There exists* $0 < \alpha_0$ *such that, given an arbitrary* $f \in H^{1/2}(\partial\Omega)$, *and any* $0 < \alpha < \alpha_0$, *the boundary value problem* (2.3) *has a unique weak solution* $E_\alpha$. *The constant* $\alpha_0$ *depends on the domains* $\{B_j\}_{j=1}^m$, $\Omega$, *the constants* $\{\mu_j, \varepsilon_j\}_{j=0}^m$, *and* $c_0$, *but is otherwise independent of the points* $\{z_j\}_{j=1}^m$. *Moreover, let* $E_0$ *denote the unique weak solution to the boundary value problem* (2.4) *corresponding to the same* $f \in H^{1/2}(\partial\Omega)$. *There exists a constant* $C$, *independent of* $\alpha$ *and* $f$, *such that*

$$\|E_\alpha - E_0\|_{H^1(\Omega)} \leqslant C\alpha^{d/2} \|f\|_{H^{1/2}(\partial\Omega)}.$$

*The constant* $C$ *depends on the domains* $\{B_j\}_{j=1}^m$, $\Omega$, *the constants* $\{\mu_j, \varepsilon_j\}_{j=0}^m$, *and* $c_0$, *but is otherwise independent of the points* $\{z_j\}_{j=1}^m$.

**Proof.** The function $E_\alpha - E_0$ is in $H_0^1(\Omega)$ and for any $v \in H_0^1(\Omega)$:

$$\begin{aligned}
a_\alpha^{\mathrm{TE}}(E_\alpha - E_0, v) &= \int_\Omega \frac{1}{\mu_\alpha} \operatorname{grad}(E_\alpha - E_0) \cdot \operatorname{grad} v - \omega^2 \int_\Omega \varepsilon_\alpha (E_\alpha - E_0) v \\
&= \sum_{j=1}^m \int_{z_j + \alpha B_j} \left[ \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \operatorname{grad} E_0 \cdot \operatorname{grad} v + \omega^2 (\varepsilon_j - \varepsilon_0) E_0 v \right].
\end{aligned}$$

Next

$$\left| \int_{z_j + \alpha B_j} \left[ \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \operatorname{grad} E_0 \cdot \operatorname{grad} v + \omega^2 (\varepsilon_j - \varepsilon_0) E_0 v \right] \right|$$

is bounded by

$$C \|E_0\|_{H^1(z_j + \alpha B_j)} \|v\|_{H^1(\Omega)}.$$

Since the inhomogeneities are bounded away from the boundary $\partial\Omega$, standard elliptic regularity results [20,45] give that

$$\|E_0\|_{W^{1,\infty}(\mathcal{B}_\alpha)} \leqslant C\|E_0\|_{H^1(\Omega)} \leqslant C\|f\|_{H^{1/2}(\partial\Omega)},$$

and so

$$\|E_0\|_{H^1(z_j+\alpha B_j)} \leqslant \|E_0\|_{W^{1,\infty}(\mathcal{B}_\alpha)}\alpha^{d/2}|B_j|^{1/2} \leqslant C\alpha^{d/2}\|f\|_{H^{1/2}(\partial\Omega)}.$$

From Lemma 2.1 it then follows immediately that

$$\|E_\alpha - E_0\|_{H^1(\Omega)} \leqslant C\alpha^{d/2}\|f\|_{H^{1/2}(\partial\Omega)},$$

exactly as desired. □

In the TM case the magnetic field $H_\alpha$ (or rather, the transversal strength) in the presence of the inhomogeneities satisfies the Helmholtz equation:

$$\operatorname{div}\left(\frac{1}{\varepsilon_\alpha}\operatorname{grad} H_\alpha\right) + \omega^2\mu_\alpha H_\alpha = 0 \quad \text{in } \Omega, \tag{2.13}$$

with the boundary condition $\frac{1}{\varepsilon_\alpha}\partial H_\alpha/\partial\nu = g \in H^{-1/2}(\partial\Omega)$ on $\partial\Omega$, where $\nu$ denotes the unit normal outward from $\Omega$ and $\partial/\partial\nu \equiv \nu\cdot\operatorname{grad}$. The magnetic field, $H_0$, in the absence of any inhomogeneities, satisfies the following equation:

$$\Delta H_0 + k^2 H_0 = 0 \quad \text{in } \Omega, \tag{2.14}$$

with $\frac{1}{\varepsilon_0}\partial H_0/\partial\nu = g$ on $\partial\Omega$. Here in order to insure well-posedness (also for the $\alpha$-dependent case for $\alpha$ sufficiently small) we shall assume that

$$\begin{aligned} &k^2 \text{ is not an eigenvalue for the operator } -\Delta \text{ in } L^2(\Omega)\\ &\quad\text{with homogeneous Neumann boundary conditions.}\end{aligned} \tag{2.15}$$

Let $a_\alpha^{\mathrm{TM}}$ denote the sesquilinear form:

$$a_\alpha^{\mathrm{TM}}(u,v) = \int_\Omega \frac{1}{\varepsilon_\alpha}\operatorname{grad} u\cdot\operatorname{grad} v - \omega^2\int_\Omega \mu_\alpha u v \tag{2.16}$$

defined on $H^1(\Omega)\times H^1(\Omega)$.

In a manner completely similar to that for the sesquilinear form $a_\alpha^{\mathrm{TE}}$ we can prove the following lemma:

**Lemma 2.2.** *Suppose* (2.1) *and* (2.15) *are satisfied, and let $a_\alpha^{\mathrm{TM}}$, $0 \leqslant \alpha$, be the sesquilinear forms defined by* (2.16)*. There exists a constant $0 < \alpha_0$, such that given any $0 \leqslant \alpha < \alpha_0$, and any bounded, conjugate-linear functional, $b$, on $H^1(\Omega)$, there is a unique $u_\alpha \in H^1(\Omega)$ which satisfies $a_\alpha^{\mathrm{TM}}(u_\alpha, v) = b(v)$ for all $v \in H^1(\Omega)$. Furthermore, there exists a constant $C$, independent of $\alpha$ and $b$, such that*

$$\|u_\alpha\|_{H^1(\Omega)} \leqslant C \sup_{v \in H^1(\Omega),\, \|v\|_{H^1(\Omega)}=1} \big|b(v)\big|.$$

As an immediate consequence of Lemma 2.2 the following holds.

**Proposition 2.2.** *Suppose* (2.1) *and* (2.15) *are satisfied. There exists $0 < \alpha_0$ such that, given an arbitrary $g \in H^{-1/2}(\partial\Omega)$, and any $0 < \alpha < \alpha_0$, the boundary value problem* (2.13) *has a unique weak solution $H_\alpha$. The constant $\alpha_0$ depends on the domains $\{B_j\}_{j=1}^m$, $\Omega$, the constants $\{\mu_j, \varepsilon_j\}_{j=0}^m$, and $c_0$, but is otherwise independent of the points $\{z_j\}_{j=1}^m$. Moreover, let $H_0$ denote the unique weak solution to the boundary value problem* (2.14) *corresponding to the same $g \in H^{-1/2}(\partial\Omega)$. There exists a constant $C$, independent of $\alpha$ and $g$, such that*

$$\|H_\alpha - H_0\|_{H^1(\Omega)} \leqslant C\alpha^{d/2}\|g\|_{H^{-1/2}(\partial\Omega)}.$$

*The constant $C$ depends on the domains $\{B_j\}_{j=1}^m$, $\Omega$, the constants $\{\mu_j, \varepsilon_j\}_{j=0}^m$, and $c_0$, but is otherwise independent of the points $\{z_j\}_{j=1}^m$.*

Let $G_D(x, y)$ be the Dirichlet Green function in $\Omega$ corresponding to a Dirac mass at the point $x$. That is, $G_D$ is the solution to

$$\begin{aligned} &-\big(\Delta_y + k^2\big)G_D(x, y) = \delta_x \quad \text{in } \Omega, \\ &G_D = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.17}$$

It is not hard to see that $G_D(x, y) = G_D(y, x)$ for any $x, y \in \Omega$ (with $x \neq y$). In terms of the special, free space Green's function

$$G^k(x, y) = \begin{cases} \dfrac{\mathrm{i}}{4} H_0^1\big(k|x-y|\big), & d = 2, \\ \dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, & d = 3, \end{cases} \tag{2.18}$$

we have $G_D(x, y) = G^k(x, y) + K(x, y)$, where $K(\cdot\,,\cdot)$ is in $C^\infty(\Omega \times \Omega)$. Furthermore, $K(x, \cdot)$ is in $C^\infty(\overline{\Omega})$ for any $x \in \Omega$, and by symmetry, $K(\cdot\,, x)$ is in $C^\infty(\overline{\Omega})$ for any $y \in \Omega$. It also proves helpful to express $G_D$ in terms of

$$G^0(x, y) = \begin{cases} -\dfrac{1}{2\pi} \log|x-y|, & d = 2, \\ \dfrac{1}{4\pi|x-y|}, & d = 3, \end{cases} \tag{2.19}$$

a free space Green's function for the Laplacian. We have:

$$\begin{aligned} &\Delta_y\big(G_D - G^0\big) + k^2\big(G_D - G^0\big) = -k^2 G^0 \quad \text{in } \Omega, \\ &\big(G_D - G^0\big) = -G^0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.20}$$

For $d = 2$ we can express $G_D$, due to (2.20), as the sum of a logarithm and a smoother function [28]. More precisely, $G_D(x, y) = -\frac{1}{2\pi}\log|x - y| + R_2(x, y)$, where for fixed $x$ the function $R_2(x, \cdot) \in W^{2,p}(\Omega)$ for any $p < \infty$. It is based on the fact that for fixed $x \in \Omega$ the function $G^0 \in C^\infty(\partial\Omega)$ and the right-hand side of (2.20) is of order $\log|x - y|$, and therefore in $L^p(\Omega)$, for any $p < \infty$ and elliptic regularity. This argument also shows that the $W^{2,p}$ norm of $R_2(x, \cdot)$ is uniformly bounded as $x$ varies over any compact subset of $\Omega$. Sobolev's Imbedding Theorem implies that (given a compact set $\mathcal{K} \subset \Omega$) there exists a constant $C$ such that

$$\big\|R_2(x, \cdot)\big\|_{L^\infty(\Omega)} + \big\|\operatorname{grad}_y R_2(x, \cdot)\big\|_{L^\infty(\Omega)} \leqslant C \quad \text{for all } x \in \mathcal{K}.$$

For $d = 3$ we can similarly express $G_D$ as the sum of $1/(4\pi|x - y|)$ and a reminder $R_3(x, \cdot) \in W^{2,p}(\Omega)$ for any $1 \leqslant p < 3$. The fact that for fixed $x$ the function $R_3(x, \cdot)$ is in $W^{2,p}(\Omega)$ for any $p < 3$ implies in particular that $R_3(x, \cdot)$ belongs to $H^1(\Omega)$ (uniformly as $x$ varies over any compact subset of $\Omega$) and is continuous on $\overline{\Omega}$.

We shall also need the Neumann Green's function. It is the solution to

$$\begin{aligned} &-\big(\Delta_y + k^2\big)G_N(x, y) = \delta_x \quad \text{in } \Omega, \\ &\frac{\partial G_N}{\partial \nu_y} = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Exactly as for the Dirichlet Green function, the following properties hold:

- $G_N(x, y) = G_N(y, x)$ for any $x, y \in \Omega$ (with $x \neq y$).
- $G_N(x, \cdot) - G^0(x, \cdot)$ belongs to $H^1(\Omega)$ (uniformly as $x$ varies over any compact subset of $\Omega$).

We can now state the following Lippman–Schwinger integral representation formulas for the electric and magnetic fields.

**Lemma 2.3.** *Suppose* (2.5) *is satisfied, the electric field $E_\alpha$ satisfies the integral representation formula*

$$\begin{aligned} E_\alpha(x) = E_0(x) &+ \sum_{j=1}^m \left(1 - \frac{\mu_0}{\mu_j}\right) \int_{z_j + \alpha B_j} \operatorname{grad} E_\alpha(y) \cdot \operatorname{grad} G_D(x, y)\, \mathrm{d}y \\ &+ k^2 \sum_{j=1}^m \left(\frac{\varepsilon_j}{\varepsilon_0} - 1\right) \int_{z_j + \alpha B_j} E_\alpha(y) G_D(x, y)\, \mathrm{d}y, \end{aligned} \tag{2.21}$$

*for all $x \in \Omega$.*

*Suppose* (2.15) *is satisfied, the magnetic field satisfies the integral representation formula*

$$H_\alpha(x) = H_0(x) + \sum_{j=1}^{m}\left(1 - \frac{\varepsilon_0}{\varepsilon_j}\right)\int_{z_j+\alpha B_j} \operatorname{grad} H_\alpha(y)\cdot \operatorname{grad} G_N(x,y)\,\mathrm{d}y$$
$$+ k^2\sum_{j=1}^{m}\left(\frac{\mu_j}{\mu_0} - 1\right)\int_{z_j+\alpha B_j} H_\alpha(y) G_N(x,y)\,\mathrm{d}y, \tag{2.22}$$

*for all $x \in \Omega$.*

**Proof.** For any $x \in \Omega$ we get, from use of (2.3) and integration by parts that

$$\int_\Omega \frac{1}{\mu_\alpha} \operatorname{grad} E_\alpha(y)\cdot \operatorname{grad} G_D(x,y)\,\mathrm{d}y - \omega^2\int_\Omega \varepsilon_\alpha E_\alpha(y) G_D(x,y)\,\mathrm{d}y = 0.$$

Therefore

$$\int_\Omega \frac{1}{\mu_0} \operatorname{grad} E_\alpha(y)\cdot \operatorname{grad} G_D(x,y)\,\mathrm{d}y - \omega^2\int_\Omega \varepsilon_0 E_\alpha(y) G_D(x,y)\,\mathrm{d}y$$
$$= \sum_{j=1}^{m}\left(\frac{1}{\mu_0} - \frac{1}{\mu_j}\right)\int_{z_j+\alpha B_j} \operatorname{grad} E_\alpha(y)\cdot \operatorname{grad} G_D(x,y)\,\mathrm{d}y$$
$$+ \omega^2\sum_{j=1}^{m}(\varepsilon_j - \varepsilon_0)\int_{z_j+\alpha B_j} E_\alpha(y) G_D(x,y)\,\mathrm{d}y. \tag{2.23}$$

We now recalculate the term in the left-hand side. Due to the common boundary condition $E_\alpha = E_0 = f$ on $\partial\Omega$, it follows from use of (2.17) that for all $x \in \Omega$ we have:

$$\int_{\partial\Omega} E_\alpha(y)\frac{\partial G_D}{\partial \nu_y}(x,y)\,\mathrm{d}s(y) = \int_{\partial\Omega} E_0(y)\frac{\partial G_D}{\partial \nu_y}(x,y)\,\mathrm{d}s(y) = -E_0(x),$$

which gives:

$$\int_\Omega \frac{1}{\mu_0} \operatorname{grad} E_\alpha(y)\cdot \operatorname{grad} G_D(x,y)\,\mathrm{d}y - \omega^2\int_\Omega \varepsilon_0 E_\alpha(y) G_D(x,y)\,\mathrm{d}y$$
$$= \frac{1}{\mu_0}\int_{\partial\Omega} E_\alpha(y)\frac{\partial G_D}{\partial \nu_y}(x,y)\,\mathrm{d}s(y) + \frac{1}{\mu_0}E_\alpha(x)$$

$$= \frac{1}{\mu_0}\big(E_\alpha(x) - E_0(x)\big) \quad \text{for all } x \in \Omega. \tag{2.24}$$

Upon insertion of (2.24) into (2.23) the Lippman–Schwinger integral representation formula (2.21) for the electric field $E_\alpha$ holds. Formula (2.22) for the magnetic field $H_\alpha$ follows in a manner completely similar to that for (2.21). The derivation of (2.21) immediately carries over; at the appropriate places we just replace the Dirichlet Green's function $G_D$ with the Neumann Green's function $G_N$. Remember that in this case we suppose $k^2$ is not an eigenvalue of the Laplacian with homogeneous Neumann boundary conditions. □

### 2.2. *Formal derivations*

In order to establish our asymptotic formulas, we may restrict, for simplicity, our attention to the case of a single inhomogeneity, i.e., the case $m = 1$. The derivation of the asymptotic expansions for any fixed number $m$ of well separated inhomogeneities (these are a fixed distance apart) follows by iteration of the arguments that we will present for the case $m = 1$. In other words, we may develop asymptotic formulas involving the difference between the fields (or the resonant/scattering frequencies) with $l$ inhomogeneities and those with $l - 1$ inhomogeneities, $l = m, \dots, 1$, and then at the end essentially form the sum of these $m$ formulas (the reference fields change, but that may be easily remedied). The derivation of each of the $m$ formulas is virtually identical. We only give the details when considering the difference between the fields corresponding to one and zero inhomogeneities. In order to further simplify notation we assume that the single inhomogeneity has the form $z + \alpha B$ and denote the electric permittivity and magnetic permeability inside $z + \alpha B$ by $\varepsilon_*$ and $\mu_*$, respectively. The condition (2.1) now translates into the condition that $\mathrm{dist}(z, \partial\Omega) > c_0$, and we note that the remainder terms in our estimates in all the following sections all depend on $c_0$ (and the shape of $B$ and $\Omega$), but are otherwise independent of the exact location of the center $z$ of the inhomogeneity. Throughout this work we use the Einstein's convention for summation notation.

For each of the problems stated above the asymptotic expansion of the solution which is uniformly valid in space is constructed by the *method of matched asymptotic expansions* for $\alpha$ small, see [16,21]. Let us focus on the derivation of asymptotic formulas for the electric field $E_\alpha$ in the TE case. The magnetic field $H_\alpha$ in the TM case has similar expansions.

To reveal the nature of the perturbations in the electric field, we introduce the local variables $\xi = (y - z)/\alpha$ and set the field $e_\alpha(\xi) = E_\alpha(z + \alpha\xi)$. We expect that $E_\alpha(x)$ will differ appreciably from $E_0(y)$ for $y$ near $z$, but it will differ little from $E_0(y)$ for $y$ far from $z$. Therefore, in the spirit of matched asymptotic expansions, we shall represent the field $E_\alpha$ by two different expansions, an *inner expansion* for $y$ near $z$, and an *outer expansion* for $y$ far from $z$. The outer expansion must begin with $E_0$, so we write:

$$E_\alpha(y) = E_0(y) + \alpha^{\tau_1} E_1(y) + \alpha^{\tau_2} E_2(y) + \cdots, \quad \text{for } |y - z| \gg \mathrm{O}(\alpha), \tag{2.25}$$

where $0 < \tau_1 < \tau_2 < \cdots$, and $E_1, E_2, \dots$ are to be found. Inserting this series into the Helmholtz equation (2.3) and observing that

$$\varepsilon_\alpha\left(\frac{y-z}{\alpha}\right) \equiv \varepsilon_0, \qquad \mu_\alpha\left(\frac{y-z}{\alpha}\right) \equiv \mu_0$$

for $|y-z| \gg \mathrm{O}(\alpha)$, we find that the outer coefficients $E_i$, $i = 1, 2, \ldots$, satisfy the simplified Helmholtz equation,

$$\left(\Delta + k^2\right)E_i = 0, \quad \text{for } |y-z| \gg \mathrm{O}(\alpha),$$

and vanish on $\partial\Omega$. Suppose (2.1) is satisfied, if a function of this kind is smooth everywhere in $\Omega$, it vanishes identically. Therefore, the outer coefficients $E_i$, $i = 1, 2, \ldots$, must have singularities at the point $z$. The behavior of the functions $E_i$ at the point $z$, and, therefore, the definition of the functions $E_i$ can be analyzed only after matching the series (2.25) to the inner expansion.

We write the inner expansion as

$$E_\alpha(z+\alpha\xi) = e_\alpha(\xi) = e_0(\xi) + \alpha e_1(\xi) + \alpha^2 e_2(\xi) + \cdots, \quad \text{for } |\xi| = \mathrm{O}(1), \tag{2.26}$$

where $e_0, e_1, e_2, \ldots$ are to be found. We assume that the functions $e_i(\xi)$ are defined not just in the domain obtained by stretching $\Omega$, but everywhere in $\mathbb{R}^d$. If we substitute the inner expansion (2.26) into the Helmholtz equation (2.3) and formally equate coefficients of $\alpha^{-2}$ and $\alpha^{-1}$ we get:

$$\operatorname{div} \frac{1}{\mu(\xi)} \operatorname{grad} e_0(\xi) = 0 \quad \text{in } \mathbb{R}^d, \tag{2.27}$$

and

$$\operatorname{div} \frac{1}{\mu(\xi)} \operatorname{grad} e_1(\xi) = 0 \quad \text{in } \mathbb{R}^d, \tag{2.28}$$

while the terms of order $\alpha^i$, $i = 0, 1, 2, \ldots$, yield

$$\operatorname{div} \frac{1}{\mu(\xi)} \operatorname{grad} e_{i+2}(\xi) = -\omega^2 \varepsilon(\xi) e_i(\xi) \quad \text{in } \mathbb{R}^d. \tag{2.29}$$

Here the stretched coefficients $\varepsilon(\xi)$ and $\mu(\xi)$ are defined by:

$$\varepsilon(\xi) = \begin{cases} \varepsilon_0, & \xi \in \mathbb{R}^d \setminus \overline{B}, \\ \varepsilon_*, & \xi \in B, \end{cases} \tag{2.30}$$

$$\mu(\xi) = \begin{cases} \mu_0, & \xi \in \mathbb{R}^d \setminus \overline{B}, \\ \mu_*, & \xi \in B. \end{cases} \tag{2.31}$$

Evidently, the functions $e_i(\xi)$ are not defined uniquely, and the question now arises as how to choose them. Thus, there is an arbitrariness in the choice of the coefficients of both the outer and the inner expansions. In order to determine the functions $E_i(y)$ and $e_i(\xi)$, we have to equate the inner and the outer expansions in some “overlap” domain within which the stretched variable $\xi$ is large and $y - z$ is small. In this domain the matching conditions are:

$$E_0(y) + \alpha^{\tau_1} E_1(y) + \alpha^{\tau_2} E_2(y) + \cdots \sim e_0(\xi) + \alpha e_1(\xi) + \alpha^2 e_2(\xi) + \cdots. \quad (2.32)$$

These matching conditions will be made more precise later on.

Inserting the outer and inner expansions into the Lippman–Schwinger integral representation formula (2.21) for $E_\alpha$ we arrive at

$$\begin{aligned} &E_0(y) + \alpha^{\tau_1} E_1(y) + \alpha^{\tau_2} E_2(y) + \cdots \\ &\quad = E_0(y) + \alpha^{d-1}\left(1 - \frac{\mu_0}{\mu_*}\right)\int_B \operatorname{grad}_\xi (e_0 + \alpha e_1 + \cdots)(\xi) \cdot \operatorname{grad} G_D(y, z + \alpha\xi)\,\mathrm{d}\xi \\ &\qquad + \alpha^d k^2\left(\frac{\varepsilon_*}{\varepsilon_0} - 1\right)\int_B (e_0 + \alpha e_1 + \cdots)(\xi) G_D(y, z + \alpha\xi)\,\mathrm{d}\xi, \end{aligned} \quad (2.33)$$

for $|y - z| \gg \mathrm{O}(\alpha)$.

Therefore, the matching conditions (2.33) imply that

$$\tau_i = d - 1 + i. \quad (2.34)$$

To illustrate this method we carry out detailed construction of the inner and outer functions $e_0, e_1, e_2, E_1,$ and $E_2$. From the terms of order $\alpha^0$, we obtain the first matching condition

$$e_0(\xi) \to E_0(z) \quad \text{as } |\xi| \to +\infty.$$

The terms of order $\alpha$ yield the second matching condition

$$e_1(\xi) - \partial_i E_0(z)\xi_i \to 0 \quad \text{as } |\xi| \to +\infty.$$

Thus, $e_0(\xi) = E_0(z)$ and $e_1(\xi) = \partial_i E_0(z)\hat{v}_{1i}^{\mu}(\xi)$, where $\hat{v}_{1i}^{\mu}$ is the unique solution to the following problem:

$$\operatorname{div} \frac{1}{\mu} \operatorname{grad} \hat{v}_{1i}^{\mu} = 0 \quad \text{in } \mathbb{R}^d, \qquad \lim_{|\xi| \to +\infty} \left(\hat{v}_{1i}^{\mu} - \xi_i\right) = 0. \quad (2.35)$$

To proceed further with the matching, we need more terms in the behavior of $\hat{v}_{1i}^{\mu}$ as $|\xi| \to +\infty$. The following lemma holds.

**Lemma 2.4.** *The unique solution $\hat{v}_{1i}^{\mu}$ to* (2.35) *satisfies the following decay estimates at infinity*:

• *For $d = 2$*

$$\hat{v}_{1i}^{\mu}(\xi) = \xi_i + \gamma_i^2 \log|\xi| + \frac{\gamma_{ij}^2 \xi_j}{|\xi|^2} + \frac{\beta_i^2}{|\xi|^2} + \frac{\beta_{ijk}^2 \xi_j \xi_k}{|\xi|^4} + \mathrm{O}\left(\frac{1}{|\xi|^3}\right), \qquad (2.36)$$

*as $|\xi| \to +\infty$, where*

$$\gamma_i^2 = 0, \qquad \gamma_{ij}^2 = \frac{1}{2\pi}\left(1 - \frac{\mu_0}{\mu_*}\right) \int_{\partial B} \xi_j \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\, \mathrm{d}s(\xi),$$

$$\beta_i^2 = -\frac{1}{4\pi}\left(1 - \frac{\mu_0}{\mu_*}\right) \int_{\partial B} |\xi|^2 \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\, \mathrm{d}s(\xi),$$

$$\beta_{ijk}^2 = -\frac{1}{2\pi}\left(1 - \frac{\mu_0}{\mu_*}\right) \int_{\partial B} \xi_j \xi_k \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\, \mathrm{d}s(\xi).$$

• *For $d = 3$*

$$\hat{v}_{1i}^{\mu}(\xi) = \xi_i + \frac{\gamma_i^3}{|\xi|} + \frac{\gamma_{ij}^3 \xi_j}{|\xi|^3} + \frac{\beta_i^3}{|\xi|^3} + \frac{\beta_{ijk}^3 \xi_j \xi_k}{|\xi|^5} + \mathrm{O}\left(\frac{1}{|\xi|^4}\right), \qquad (2.37)$$

*as $|\xi| \to +\infty$, where*

$$\gamma_i^3 = 0, \qquad \gamma_{ij}^3 = \frac{1}{4\pi}\left(1 - \frac{\mu_0}{\mu_*}\right) \int_{\partial B} \xi_j \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\, \mathrm{d}s(\xi),$$

$$\beta_i^3 = -\frac{1}{8\pi}\left(1 - \frac{\mu_0}{\mu_*}\right) \int_{\partial B} |\xi|^2 \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\, \mathrm{d}s(\xi),$$

$$\beta_{ijk}^3 = \frac{3}{16\pi}\left(1 - \frac{\mu_0}{\mu_*}\right) \int_{\partial B} \xi_j \xi_k \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\, \mathrm{d}s(\xi).$$

**Proof.** Before proving this lemma let us introduce one more notation. For a function $u$ defined on $\mathbb{R}^d$ we denote:

$$\frac{\partial u}{\partial \nu}\bigg|_{\pm}(\xi) = \lim_{t \to 0^+} \operatorname{grad} u\big(\xi \pm t\nu(\xi)\big) \cdot \nu(\xi), \quad \xi \in \partial B,$$

if the limit exists, where $\nu(\xi)$ is the outward unit normal to $\partial B$ at $\xi$. To obtain the behaviors (2.36) and (2.37) as $|\xi| \to +\infty$ we express $\hat{v}_{1i}^{\mu}$ for $\xi \in \mathbb{R}^d \setminus \overline{B}$ as follows:

$$\hat{v}_{1i}^{\mu}(\xi) = \xi_i - \int_{\partial B} \left( \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{+} - \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-} \right)(\xi') G^0(\xi, \xi')\, \mathrm{d}s(\xi')$$
$$= \xi_i + \left(1 - \frac{\mu_0}{\mu_*}\right) \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi') G^0(\xi, \xi')\, \mathrm{d}s(\xi'). \tag{2.38}$$

Since the free space Green's function $G^0(\xi, \xi')$ has the following behavior (decay) at infinity:

- For $d = 2$

$$G^0(\xi, \xi') = -\frac{1}{2\pi} \log|\xi| + \frac{\xi_j \xi_j'}{2\pi|\xi|^2} - \frac{(\xi_j')^2}{4\pi|\xi|^2} - \frac{(\xi \cdot \xi')^2}{2\pi|\xi|^4} + \mathrm{O}\left(\frac{1}{|\xi|^3}\right),$$

  as $|\xi| \to +\infty$.
- For $d = 3$

$$G^0(\xi, \xi') = \frac{1}{4\pi|\xi|} + \frac{\xi_j \xi_j'}{4\pi|\xi|^3} - \frac{(\xi_j')^2}{8\pi|\xi|^3} + \frac{3(\xi \cdot \xi')^2}{16\pi|\xi|^5} + \mathrm{O}\left(\frac{1}{|\xi|^4}\right) \quad \text{as } |\xi| \to +\infty,$$

by inserting this into the integral representation formula (2.38) and assuming for a moment that $\partial \hat{v}_{1i}^{\mu}/\partial\nu|_{-} \in L^\infty(\partial B)$ we immediately obtain (2.36) and (2.37) due to the fact that

$$\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\, \mathrm{d}s(\xi) = 0.$$

We now prove the uniqueness of a solution $\hat{v}_{1i}^{\mu}$ to (2.35) and justify that $\partial \hat{v}_{1i}^{\mu}/\partial\nu|_{-} \in L^\infty(\partial B)$. Let $w$ be the difference of two solutions, so that

$$\operatorname{div} \frac{1}{\mu} \operatorname{grad} w = 0 \quad \text{in} \quad \mathbb{R}^d \quad \text{and} \quad w(\xi) = \mathrm{O}\big(|\xi|^{-d+1}\big) \quad \text{as } |\xi| \to +\infty.$$

Integration by parts yields the energy identity,

$$\int_{|\xi|<R} \frac{1}{\mu} |\operatorname{grad} w|^2 = \int_{|\xi|=R} \frac{1}{\mu} w \frac{\partial w}{\partial \nu}.$$

Now let $R \to +\infty$; we have $w\, \partial w/\partial\nu = \mathrm{O}(R^{-2d+1})$ for $|\xi| = R$, so that

$$\int_{I R^d} \frac{1}{\mu} |\operatorname{grad} w|^2 = 0.$$

This implies that $w$ is constant in $\mathbb{R}^d$, and, in fact, $w = 0$ in $\mathbb{R}^d$ because $w \to 0$ as $|\xi| \to +\infty$.

The existence of $\hat{v}_{1i}^{\mu}$ can be established using double layer potentials with suitably chosen densities. Fairly simple manipulations show that $\hat{v}_{1i}^{\mu}|_{\partial B}$ satisfies the integral equation:

$$\frac{1}{2}\left(1+\frac{\mu_0}{\mu_*}\right)\hat{v}_{1i}^{\mu}|_{\partial B}(\xi) + \left(\frac{\mu_0}{\mu_*}-1\right)\int_{\partial B} \hat{v}_{1i}^{\mu}|_{\partial B}(\xi')\frac{\partial G^0}{\partial \nu(\xi')}(\xi,\xi')\,\mathrm{d}s(\xi') = \xi_i, \quad \xi \in \partial B. \tag{2.39}$$

The uniqueness of a solution to (2.35) ensures that the above integral equation has at most one solution. Since the double layer potential,

$$\hat{v}_{1i}^{\mu}|_{\partial B} \mapsto \int_{\partial B} \hat{v}_{1i}^{\mu}|_{\partial B}(\xi')\frac{\partial G^0}{\partial \nu(\xi')}(\xi,\xi')\,\mathrm{d}s(\xi'),$$

is continuous from $L^2(\partial B)$ to $H^1(\partial B)$, Fredholm alternative guarantees that $\hat{v}_{1i}^{\mu}|_{\partial B}$ is the only solution to the above integral equation. Analogously to (2.39), it follows from classical potential theory that

$$\frac{1}{2}\frac{\mu_0/\mu_*+1}{\mu_0/\mu_*-1}\frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi) + \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi')\frac{\partial G^0}{\partial \nu(\xi)}(\xi,\xi')\,\mathrm{d}s(\xi') = \frac{1}{\mu_0/\mu_*-1}\xi_i, \quad \xi \in \partial B.$$

Denote

$$K_B^*(\varphi)(\xi) = \int_{\partial B} \varphi(\xi')\frac{\partial G^0}{\partial \nu(\xi)}(\xi,\xi')\,\mathrm{d}s(\xi').$$

From [25,46] we know that the operator $\lambda I - K_B^*$ is invertible on $L_0^{\infty}(\partial B) = \{\varphi \in L^{\infty}(\partial B)\colon \int_{\partial B}\varphi = 0\}$. Thus, $\partial \hat{v}_{1i}^{\mu}/\partial\nu|_{-} \in L_0^{\infty}(\partial B)$. This completes the proof. □

We now use $e_1(\xi) = \partial_i E_0(z)\hat{v}_{1i}^{\mu}(\xi)$ in (2.32) and set $\xi = (y-z)/\alpha$. Then the second term in the far field form of $\alpha\partial_i E_0(z)\hat{v}_{1i}^{\mu}(\xi)$ becomes

$$\alpha^d \gamma_{ij}^d \partial_i E_0(z)\frac{x_j}{|x|^{d-1}},$$

which must be matched with the second term $\alpha^{\tau_1}E_1(x)$ on the left-hand side of (2.32). Therefore, $\tau_1$ must be equal to $d$ which is in accordance with (2.34).

The asymptotic formula (2.33) yields:

$$E_1(y) = \left(1 - \frac{\mu_0}{\mu_*}\right)\partial_i E_0(z)\left(\int_B \operatorname{grad} \hat{v}^{\mu}_{1i}(\xi)\,\mathrm{d}\xi\right) \cdot \operatorname{grad} G_D(y,z)$$
$$+ k^2\left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right)|B| E_0(z) G_D(y,z), \quad |y - z| \gg \mathrm{O}(\alpha). \tag{2.40}$$

Since

$$\int_B \left(\operatorname{grad} \hat{v}^{\mu}_{1i}(\xi)\,\mathrm{d}\xi\right) \cdot \operatorname{grad} G_D(y,z) = \int_{\partial B} \frac{\partial \hat{v}^{\mu}_{1i}}{\partial \nu}\bigg|_- (\xi)\xi_j\,\mathrm{d}s(\xi)\partial_j G_D(y,z)$$

formula (2.40) can be rewritten in the more convenient form:

$$E_1(y) = \left(1 - \frac{\mu_0}{\mu_*}\right)\partial_i E_0(z) M^{1,1}_{ij} \partial_j G_D(y,z)$$
$$+ k^2\left(\frac{\varepsilon_*}{\varepsilon_0} - 1\right)|B| E_0(z) G_D(y,z), \quad |y - z| \gg \mathrm{O}(\alpha), \tag{2.41}$$

where the generalized polarization tensor $M^{1,1}$ of order $(1,1)$ is defined by:

$$M^{1,1}_{ij} = \int_{\partial B} \frac{\partial \hat{v}^{\mu}_{1i}}{\partial \nu}\bigg|_- (\xi)\xi_j\,\mathrm{d}s(\xi).$$

The reader is referred to Section 3 for a detailed study of the properties of $M^{1,1}$.

Let $\hat{v}^{\mu,\varepsilon}_{2ij}$ be the unique solution to the following problem:

$$\begin{aligned}
&\operatorname{div} \frac{1}{\mu} \operatorname{grad} \hat{v}^{\mu,\varepsilon}_{2ij} = \frac{1}{\mu_0}\frac{\varepsilon(\xi)}{\varepsilon_0}\delta_{ij} \quad \text{in } \mathbb{R}^d,\\
&\lim_{|\xi| \to +\infty}\left(\hat{v}^{\mu,\varepsilon}_{2ij} - \frac{1}{2}\xi_i\xi_j - \delta_{ij}\frac{1}{2\pi}|B|\left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right)\log|\xi|\right) = 0 \quad \text{if } d = 2,\\
&\lim_{|\xi| \to +\infty}\left(\hat{v}^{\mu,\varepsilon}_{2ij} - \frac{1}{2}\xi_i\xi_j\right) = 0 \quad \text{if } d = 3.
\end{aligned} \tag{2.42}$$

The following holds.

**Lemma 2.5.** *The unique solution* $\hat{v}^{\mu,\varepsilon}_{2ij}$ *to* (2.42) *satisfies the following decay estimates at infinity*:

- *For* $d = 2$

$$\hat{v}_{2ij}^{\mu,\varepsilon}(\xi) = \frac{1}{2}\xi_i\xi_j - \delta_{ij}\frac{1}{2\pi}|B|\left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right)\log|\xi| + \frac{\xi_j}{|\xi|^2} + \mathrm{O}\left(\frac{1}{|\xi|^2}\right), \quad \text{as } |\xi| \to +\infty. \tag{2.43}$$

- *For* $d = 3$

$$\hat{v}_{2ij}^{\mu,\varepsilon}(\xi) = \frac{1}{2}\xi_i\xi_j + \delta_{ij}\frac{1}{4\pi}|B|\left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right)\frac{1}{|\xi|} + \frac{1}{4\pi}\left(\delta_{ij}\left(1 - \frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}\right)\int_B \xi_k'\,\mathrm{d}\xi' + \left(1 - \frac{\mu_0}{\mu_*}\right)\int_{\partial B} \xi_k' \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_-(\xi')\,\mathrm{d}s(\xi')\right)\frac{\xi_k}{|\xi|^3} + \mathrm{O}\left(\frac{1}{|\xi|^3}\right) \quad \text{as } |\xi| \to +\infty. \tag{2.44}$$

**Proof.** The uniqueness of $\hat{v}_{2ij}^{\mu,\varepsilon}$ may be proven using an energy argument in the same way as done in Lemma 2.4.

The existence of $\hat{v}_{2ij}^{\mu,\varepsilon}$ is most easily established by noting that $\hat{v}_{2ij}^{\mu,\varepsilon}|_{\partial B}$ satisfies the integral equation:

$$\frac{1}{2}\left(1 + \frac{\mu_0}{\mu_*}\right)\hat{v}_{2ij}^{\mu,\varepsilon}|_{\partial B}(\xi) + \left(\frac{\mu_0}{\mu_*} - 1\right)\int_{\partial B} \hat{v}_{2ij}^{\mu,\varepsilon}|_{\partial B}(\xi')\frac{\partial G^0}{\partial \nu(\xi')}(\xi,\xi')\,\mathrm{d}s(\xi') = \frac{1}{2}\xi_i\xi_j + \delta_{ij}\left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right)\int_B G^0(\xi,\xi')\,\mathrm{d}\xi', \quad \xi \in \partial B.$$

The decay estimates follow immediately from the integral representation formula:

$$\hat{v}_{2ij}^{\mu,\varepsilon}(\xi) = \frac{1}{2}\xi_i\xi_j - \delta_{ij}\left(\frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0} - 1\right)\int_B G^0(\xi,\xi')\,\mathrm{d}\xi' + \left(1 - \frac{\mu_0}{\mu_*}\right)\int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_-(\xi')G^0(\xi,\xi')\,\mathrm{d}s(\xi'),$$

for $\xi \in \mathbb{R}^d \setminus \overline{B}$, and the fact that

$$\int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_-(\xi)\,\mathrm{d}s(\xi) = \frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}|B|\delta_{ij}. \qquad \square$$

Since $e_2(\xi)$ satisfies

$$\operatorname{div} \frac{1}{\mu(\xi)} \operatorname{grad} e_2(\xi) = -\omega^2 \varepsilon(\xi) e_0(\xi) = -\omega^2 \varepsilon(\xi) E_0(z) \quad \text{in } \mathbb{R}^d,$$

the matching conditions (2.32) that are:

$$e_2(\xi) \sim \begin{cases} \frac{1}{2}\xi_i \xi_j \partial_{ij}^2 E_0(z) + \frac{1}{2\pi} k^2 |B| \left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right) \log|\xi| E_0(z) & \text{if } d = 2, \\ \frac{1}{2}\xi_i \xi_j \partial_{ij}^2 E_0(z) & \text{if } d = 3, \end{cases}$$

as $|\xi| \to +\infty$, define uniquely $e_2(\xi)$. From the definition of $\hat{v}_{2ij}^{\mu,\varepsilon}$ it immediately follows that

$$e_2(\xi) = \partial_{ij} E_0(z) \hat{v}_{2ij}^{\mu,\varepsilon}(\xi).$$

The asymptotic formula (2.33) therefore yields:

$$\begin{aligned} E_2(y) = {} & \left(1 - \frac{\mu_0}{\mu_*}\right) \left[ \partial_{ij}^2 E_0(z) \left( \int_B \operatorname{grad} \hat{v}_{2ij}^{\mu,\varepsilon}(\xi)\, \mathrm{d}\xi \right) \cdot \operatorname{grad} G_D(y,z) \right. \\ & \left. + \partial_i E_0(z) \left( \int_B \xi_j \partial_k \hat{v}_{1i}^{\mu}(\xi)\, \mathrm{d}\xi \right) \partial_{jk}^2 G_D(y,z) \right] \\ & + k^2 \left( \frac{\varepsilon_*}{\varepsilon_0} - 1 \right) \left[ \partial_i E_0(z) \left( \int_B \hat{v}_{1i}^{\mu}(\xi)\, \mathrm{d}\xi \right) G_D(y,z) \right. \\ & \left. + E_0(z) \left( \int_B \xi_j \, \mathrm{d}\xi \right) \partial_j G_D(y,z) \right], \quad |y - z| \gg \mathrm{O}(\alpha). \end{aligned} \tag{2.45}$$

Define the generalized polarization tensors of order $(1,2)$ and $(2,1)$ by:

$$M_{ijk}^{1,2} = \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-} (\xi) \xi_j \xi_k \, \mathrm{d}s(\xi) \quad \text{and} \quad M_{ijk}^{2,1} = \int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_{-} (\xi) \xi_k \, \mathrm{d}s(\xi).$$

The reader is referred to Section 3 for the study of these tensors. Since

$$\int_B \left( \xi_j \partial_k \hat{v}_{1i}^{\mu}(\xi) + \xi_k \partial_j \hat{v}_{1i}^{\mu}(\xi) \right) \mathrm{d}\xi = \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-} (\xi) \xi_j \xi_k \, \mathrm{d}s(\xi)$$

and

$$
\left(\int_B \operatorname{grad} \hat{v}_{2ij}^{\mu,\varepsilon}(\xi)\,\mathrm{d}\xi\right)\cdot \operatorname{grad} G_D(y,z)
$$
$$
= \left(\int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_{-}(\xi)\xi_k\,\mathrm{d}s(\xi) bR\right)\partial_k G_D(y,z) - \frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}\left(\int_B \xi_k\,\mathrm{d}\xi\right)\delta_{ij}\partial_k G_D(y,z)
$$

formula (2.45) becomes:

$$
\begin{aligned}
E_2(y) = {} & \left(1-\frac{\mu_0}{\mu_*}\right)\left[\partial_{ij}^2 E_0(z) M_{ijk}^{2,1}\partial_k G_D(y,z) + \partial_i E_0(z) M_{ijk}^{1,2}\partial_{jk}^2 G_D(y,z)\right] \\
& + k^2\left(\frac{\varepsilon_*}{\varepsilon_0}-1\right)\Bigg[\partial_i E_0(z)\left(\int_B \hat{v}_{1i}^{\mu}(\xi)\,\mathrm{d}\xi\right) G_D(y,z) \\
& + E_0(z)\left(\int_B \xi_j\,\mathrm{d}\xi\right)\partial_j G_D(y,z)\Bigg], \quad |y-z| \gg \mathrm{O}(\alpha).
\end{aligned} \tag{2.46}
$$

Additional terms in the inner and outer expansions can be obtained by analyzing the higher-order matching conditions in (2.32).

It is well known that the inner and outer expansions are not valid uniformly in $y$. To obtain an asymptotic expansion of the fields as $\alpha \to 0$ that is valid uniformly in space, we construct the composite expansion of the method of matched asymptotic expansions. Thus, adding the outer and inner expansions and subtracting out the common part, we formally obtain the following *uniform expansions*: for all $y \in \Omega$:

$$
\begin{aligned}
E_\alpha(y) = {} & E_0(y) + \alpha\Bigg[e_1\left(\frac{y-z}{\alpha}\right) - \frac{(y-z)}{\alpha}\cdot \operatorname{grad} E_0(z) \\
& - \frac{\alpha}{2\pi}\left(1-\frac{\mu_0}{\mu_*}\right)\left(\int_{\partial B}\frac{\partial e_1}{\partial \nu}\bigg|_{-}(\xi)\xi_i\,\mathrm{d}s(\xi)\right)\frac{(y-z)_i}{|y-z|^2}\Bigg] \\
& + \alpha^2\Bigg[e_2\left(\frac{y-z}{\alpha}\right) - \frac{1}{2}\frac{(y-z)_i}{\alpha}\frac{(y-z)_j}{\alpha}\partial_{ij}^2 E_0(z) \\
& - \frac{1}{2\pi}k^2\left(1-\frac{\varepsilon_*}{\varepsilon_0}\right)|B| E_0(z)\log\left(\frac{|y-z|}{\alpha}\right)\Bigg] + \alpha^2 E_1(y) + \mathrm{o}\big(\alpha^2\big),
\end{aligned} \tag{2.47}
$$

for $d = 2$,

$$
\begin{aligned}
E_\alpha(y) = {} & E_0(y) + \alpha\left[e_1\left(\frac{y-z}{\alpha}\right) - \frac{(y-z)}{\alpha}\cdot \operatorname{grad} E_0(z)\right] \\
& + \alpha^2\left[e_2\left(\frac{y-z}{\alpha}\right) - \frac{1}{2}\frac{(y-z)_i}{\alpha}\frac{(y-z)_j}{\alpha}\partial_{ij}^2 E_0(z)\right] + \mathrm{o}\big(\alpha^2\big),
\end{aligned} \tag{2.48}
$$

for $d = 3$. Finally, inserting the inner expansions into the Lippman–Schwinger integral representation formula (2.21) for $E_\alpha$ we may conclude that the following pointwise asymptotic expansion on $\partial\Omega$ holds for $d = 2, 3$:

$$\frac{\partial E_\alpha(y)}{\partial\nu(y)} = \frac{\partial E_0(y)}{\partial\nu(y)}) + \alpha^d \frac{\partial E_1(y)}{\partial\nu(y)} + \alpha^{d+1} \frac{\partial E_2(y)}{\partial\nu(y)} + \mathrm{o}\big(\alpha^{d+1}\big), \tag{2.49}$$

where the remainder $\mathrm{o}(\alpha^{d+1})$ is independent of $x \in \partial\Omega$.

## 2.3. *Proof of the asymptotic expansions*

In this section we shall show that the fields $E_0, E_1, \hat{v}^\mu_{1i}$, and $\hat{v}^{\mu,\varepsilon}_{2ij}$ allow us to obtain approximations (valid uniformly in space) to $E_\alpha$ of order $\alpha^{3-\delta}$ for $d = 2$ and $\alpha^3$ for $d = 3$. The pointwise asymptotic expansion (2.49) will rigorously follow from these uniform approximations. Without loss of generality, we restrict ourselves to the case of one inhomogeneity.

Introduce the auxiliary fields:

$$\mathcal{E}_1(\xi) = \begin{cases} \left(\hat{v}^\mu_{1i}(\xi) - \xi_i - \dfrac{1}{2\pi}\left(1 - \dfrac{\mu_0}{\mu_*}\right) M^{1,1}_{ij} \dfrac{\xi_j}{|\xi|^2}\right)\partial_i E_0(z) & \text{for } d = 2, \\ \big(\hat{v}^\mu_{1i}(\xi) - \xi_i\big)\partial_i E_0(z) & \text{for } d = 3, \end{cases}$$

and

$$\mathcal{E}_2(\xi) = \begin{cases} \left(\hat{v}^{\mu,\varepsilon}_{2ij}(\xi) - \dfrac{1}{2}\xi_i\xi_j\right)\partial^2_{ij} E_0(z) - \dfrac{1}{2\pi}k^2\left(1 - \dfrac{\varepsilon_*}{\varepsilon_0}\right)|B| E_0(z)\log|\xi| & \text{for } d = 2, \\ \left(\hat{v}^{\mu,\varepsilon}_{2ij}(\xi) - \dfrac{1}{2}\xi_i\xi_j\right)\partial^2_{ij} E_0(z) & \text{for } d = 3. \end{cases}$$

The main results proven in this section are the following:

**Theorem 2.1.** *Suppose* (2.1) *and* (2.5) *are satisfied. There exists a constant $C$, independent of $\alpha$ and $f$, such that*:

- *For $d = 2$*

$$\left\| E_\alpha(x) - E_0(x) - \alpha\mathcal{E}_1\left(\frac{x-z}{\alpha}\right) - \alpha^2 E_1(x) - \alpha^2\mathcal{E}_2\left(\frac{x-z}{\alpha}\right)\right\|_{H^1(\Omega)} \leqslant C\alpha^{3-\delta}\|f\|_{H^{1/2}(\partial\Omega)}, \tag{2.50}$$

  *for any fixed $\delta > 0$.*
- *For $d = 3$*

$$\left\| E_\alpha(x) - E_0(x) - \alpha \mathcal{E}_1\left(\frac{x-z}{\alpha}\right) - \alpha^2 \mathcal{E}_2\left(\frac{x-z}{\alpha}\right) \right\|_{H^1(\Omega)} \leqslant C\alpha^3 \|f\|_{H^{1/2}(\partial\Omega)}. \tag{2.51}$$

- *For $x \in \partial\Omega$ the following pointwise asymptotic expansion holds*:

$$\frac{\partial E_\alpha(x)}{\partial \nu(x)} = \frac{\partial E_0(x)}{\partial \nu(x)} + \alpha^d \frac{\partial E_1(x)}{\partial \nu(x)} + \alpha^{d+1} \frac{\partial E_2(x)}{\partial \nu(x)} + \begin{cases} \mathrm{O}(\alpha^{4-\delta}) & \text{if } d=2, \\ \mathrm{O}(\alpha^5) & \text{if } d=3, \end{cases} \tag{2.52}$$

*for any fixed $\delta > 0$, where the remainders $\mathrm{O}(\alpha^{4-\delta})$ and $\mathrm{O}(\alpha^5)$, and all their derivatives, are uniformly bounded on $\partial\Omega$ by $C\alpha^{4-\delta}\|f\|_{H^{1/2}(\partial\Omega)}$ and $C\alpha^5\|f\|_{H^{1/2}(\partial\Omega)}$, respectively.*

*The constant $C$ depends on the domains $B$, $\Omega$, the constants $\mu_0, \varepsilon_0, \mu_*, \varepsilon_*$, and $c_0$.*

**Proof.** Since $\operatorname{dist}(z, \partial\Omega) \geqslant c_0 > 0$, the pointwise asymptotic expansions (2.52) are immediately obtained from the Lippman–Schwinger integral representation formula (2.21) as a direct consequence of the $H^1$-estimates (2.50) and (2.51).

It will turn out to be convenient to introduce slight variations of $\hat{v}_{1i}^\mu$ and $\hat{v}_{2ij}^{\mu,\varepsilon}$, that satisfy specified boundary conditions on the boundary of the domain $\widetilde{\Omega} = (\Omega - z)/\alpha$. Let $v_{1i}^*$ and $v_{2ij}^*$ be defined as the unique solution to the boundary value problem:

$$\begin{aligned}
&\operatorname{div} \frac{1}{\mu} \operatorname{grad} \hat{v}_{1i}^*(\xi) = 0 \quad \text{in } \widetilde{\Omega}, \\
&\hat{v}_{1i}^*(\xi) - \xi_i - \frac{1}{2\pi}\left(1 - \frac{\mu_0}{\mu_*}\right)\left(\int_{\partial B} \xi_j' \frac{\partial \hat{v}_{1i}^\mu}{\partial \nu}\bigg|_-(\xi')\,\mathrm{d}s(\xi')\right)\frac{\xi_j}{|\xi|^2} = 0 \quad \text{on } \partial\widetilde{\Omega} \text{ if } d=2, \\
&\hat{v}_{1i}^*(\xi) - \xi_i = 0 \quad \text{on } \partial\widetilde{\Omega} \text{ if } d=3,
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{div} \frac{1}{\mu} \operatorname{grad} \hat{v}_{2ij}^*(\xi) = \frac{1}{\mu_0}\frac{\varepsilon(\xi)}{\varepsilon_0}\delta_{ij} \quad \text{in } \widetilde{\Omega}, \\
&\hat{v}_{2ij}^*(\xi) - \frac{1}{2}\xi_i\xi_j - \delta_{ij}\frac{1}{2\pi}|B|\left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right)\log|\xi| = 0 \quad \text{on } \partial\widetilde{\Omega} \quad \text{if } d=2, \\
&\hat{v}_{2ij}^*(\xi) - \frac{1}{2}\xi_i\xi_j = 0 \quad \text{on } \partial\widetilde{\Omega} \text{ if } d=3,
\end{aligned}$$

respectively.

We begin by establishing the following lemma which estimates the differences $\hat{v}_{1i}^\mu - v_{1i}^*$ and $\hat{v}_{2ij}^{\mu,\varepsilon} - v_{2ij}^*$.

**Lemma 2.6.** *With $v_{1i}^*$ and $v_{2ij}^*$ as introduced above we have*:

$$\left\|\left(\hat{v}_{1i}^{\mu}-v_{1i}^*\right)\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)}+\left\|\mathrm{grad}_\xi\left(\hat{v}_{1i}^{\mu}-v_{1i}^*\right)\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)}\leqslant C\alpha^2, \quad (2.53)$$

*and*

$$\left\|\left(\hat{v}_{2ij}^{\mu,\varepsilon}-v_{2ij}^*\right)\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)}+\left\|\mathrm{grad}_\xi\left(\hat{v}_{2ij}^{\mu,\varepsilon}-v_{2ij}^*\right)\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)}\leqslant C\alpha. \quad (2.54)$$

**Proof.** Set $w(\xi)=(\hat{v}_{1i}^{\mu}-v_{1i}^*)(\xi)$. We have:

$$\begin{aligned}\int_{\widetilde{\Omega}}\frac{1}{\mu(\xi)}\left|\mathrm{grad}_\xi\, w(\xi)\right|^2\mathrm{d}\xi &= \int_{\partial\widetilde{\Omega}}\frac{1}{\mu_0}\frac{\partial w}{\partial\nu}(\xi)w(\xi)\,\mathrm{d}s(\xi)\\ &\leqslant C\left\|\frac{\partial w}{\partial\nu(\xi)}\right\|_{H^{-1/2}(\partial\widetilde{\Omega})}\|w\|_{H^{1/2}(\partial\widetilde{\Omega})}.\end{aligned}$$

By the definition of the $H^{-1/2}$ norm we may write:

$$\begin{aligned}\left\|\frac{\partial w}{\partial\nu(\xi)}\right\|_{H^{-1/2}(\partial\widetilde{\Omega})} &= \sup_{\|\varphi\|_{H^{1/2}(\partial\widetilde{\Omega})}=1}\left|\int_{\partial\widetilde{\Omega}}\frac{\partial w}{\partial\nu}(\xi)\varphi(\xi)\,\mathrm{d}s(\xi)\right|\\ &= \alpha^{-d+1}\sup_{\|\varphi\|_{H^{1/2}(\partial\widetilde{\Omega})}=1}\left|\int_{\partial\Omega}\frac{\partial w}{\partial\nu(\xi)}\left(\frac{x-z}{\alpha}\right)\varphi\left(\frac{x-z}{\alpha}\right)\mathrm{d}s(x)\right|\\ &\leqslant \alpha^{-d+1}\left\|\frac{\partial w}{\partial\nu(\xi)}\left(\frac{x-z}{\alpha}\right)\right\|_{H^{-1/2}(\partial\Omega)}\sup_{\|\varphi\|_{H^{1/2}(\partial\widetilde{\Omega})}=1}\left\|\varphi\left(\frac{x-z}{\alpha}\right)\right\|_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

But we have, by the change of variables, $x=\alpha\xi+z$

$$\left\|\varphi\left(\frac{x-z}{\alpha}\right)\right\|_{H^{1/2}(\partial\Omega)}=\alpha^{d/2-1/2}\|\varphi(\xi)\|_{H^{1/2}(\partial\widetilde{\Omega})},$$

for all $\varphi\in H^{1/2}(\partial\widetilde{\Omega})$ from which it follows that

$$\left\|\frac{\partial w}{\partial\nu(\xi)}\right\|_{H^{-1/2}(\partial\widetilde{\Omega})}\leqslant\alpha^{-d/2+1/2}\left\|\frac{\partial w}{\partial\nu(\xi)}\left(\frac{x-z}{\alpha}\right)\right\|_{H^{-1/2}(\partial\Omega)},$$

and so,

$$\int_{\widetilde{\Omega}}\frac{1}{\mu(\xi)}\left|\mathrm{grad}_\xi\, w(\xi)\right|^2\mathrm{d}\xi\leqslant C\left\|\frac{\partial w}{\partial\nu(\xi)}\left(\frac{x-z}{\alpha}\right)\right\|_{H^{-1/2}(\partial\Omega)}\left\|w\left(\frac{x-z}{\alpha}\right)\right\|_{H^{1/2}(\partial\Omega)}. \quad (2.55)$$

The decay at infinity of $\hat{v}_{1i}^{\mu}(\xi)$ gives:

$$
\begin{aligned}
&\left\| w\left(\frac{x-z}{\alpha}\right)\right\|_{H^{1/2}(\partial\Omega)} \\
&\leqslant C \begin{cases} \left\| \hat{v}_{1i}^{\mu}\left(\frac{x-z}{\alpha}\right) - \left(\frac{x-z}{\alpha}\right)_i - \frac{\alpha}{2\pi}\left(1-\frac{\mu_0}{\mu_*}\right) M_{ij}^{1,1}\frac{(x-z)_j}{|x-z|^2}\right\|_{H^1(\mathbb{R}^2\setminus\overline{\Omega})} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } d=2, \\ \left\| \hat{v}_{1i}^{\mu}\left(\frac{x-z}{\alpha}\right) - \left(\frac{x-z}{\alpha}\right)_i\right\|_{H^1(\mathbb{R}^3\setminus\overline{\Omega})} \quad \text{if } d=3, \end{cases} \\
&\leqslant C\alpha^2,
\end{aligned}
$$

and therefore,

$$
\int_{\widetilde{\Omega}} \frac{1}{\mu(\xi)}\left|\operatorname{grad}_\xi w(\xi)\right|^2 \mathrm{d}\xi \leqslant C\alpha^2 \left\| \frac{\partial w}{\partial \nu(\xi)}\left(\frac{x-z}{\alpha}\right)\right\|_{H^{-1/2}(\partial\Omega)}. \tag{2.56}
$$

We also have, by the change of variables, $x = \alpha\xi + z$,

$$
\int_{\Omega} \frac{1}{\mu_\alpha}\left(\frac{x-z}{\alpha}\right)\left|\operatorname{grad}_\xi w\left(\frac{x-z}{\alpha}\right)\right|^2 \mathrm{d}x = \alpha^d \int_{\widetilde{\Omega}} \frac{1}{\mu(\xi)}\left|\operatorname{grad}_\xi w(\xi)\right|^2 \mathrm{d}\xi, \tag{2.57}
$$

and

$$
\int_{\Omega} \left| w\left(\frac{x-z}{\alpha}\right)\right|^2 \mathrm{d}x = \alpha^d \int_{\widetilde{\Omega}} \left|w(\xi)\right|^2 \mathrm{d}\xi. \tag{2.58}
$$

The term $\|w((x-z)/\alpha)\|_{L^2(\Omega)}$ can be estimated by the Poincaré inequality as follows:

$$
\left\| w\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)} \leqslant C\left(\left\|\operatorname{grad}_x w\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)} + \left|\int_{\partial\Omega} w\left(\frac{x-z}{\alpha}\right)\mathrm{d}s(x)\right|\right)
$$

with

$$
\left|\int_{\partial\Omega} w\left(\frac{x-z}{\alpha}\right)\mathrm{d}(x)\right| \leqslant C\alpha^2.
$$

Therefore

$$
\left\| w\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)} \leqslant C\alpha^2 + C\alpha^{d/2-1}\|\operatorname{grad}_\xi w\|_{L^2(\widetilde{\Omega})}.
$$

Since $\Delta w((x-z)/\alpha)=0$ in a neighborhood of $\partial\Omega$ we may verify that the following estimate holds:

$$\left\| \frac{\partial w}{\partial \nu(\xi)}\left(\frac{x-z}{\alpha}\right)\right\|_{H^{-1/2}(\partial\Omega)} \leqslant C\alpha\left(\left\| w\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)} + \left\|\operatorname{grad}_x w\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)}\right)$$
$$\leqslant C\alpha^3 + C\alpha^{d/2}\|\operatorname{grad}_\xi w\|_{L^2(\widetilde{\Omega})}. \tag{2.59}$$

By a combination of (2.56)–(2.58) we now get

$$\left\|\left(\hat{v}_{1i}^{\mu} - v_{1i}^*\right)\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)} + \left\|\operatorname{grad}_\xi\left(\hat{v}_{1i}^{\mu} - v_{1i}^*\right)\left(\frac{x-z}{\alpha}\right)\right\|_{L^2(\Omega)} \leqslant C\alpha^2,$$

exactly as desired. Estimate (2.54) can be established in the exactly same manner and so, we omit its proof. □

We return to the proof of Theorem 2.1. According to the previous lemma, it suffices to establish the estimates

$$\left\| E_\alpha(x) - E_0(x) - \alpha\mathcal{E}_1^*\left(\frac{x-z}{\alpha}\right) - \alpha^2 E_1(x) - \alpha^2\mathcal{E}_2^*\left(\frac{x-z}{\alpha}\right)\right\|_{H^1(\Omega)}$$
$$\leqslant C\alpha^{3-\delta}\|f\|_{H^{1/2}(\partial\Omega)}, \quad \text{for } d=2,$$
$$\left\| E_\alpha(x) - E_0(x) - \alpha\mathcal{E}_1^*\left(\frac{x-z}{\alpha}\right) - \alpha^2\mathcal{E}_2^*\left(\frac{x-z}{\alpha}\right)\right\|_{H^1(\Omega)}$$
$$\leqslant C\alpha^{3}\|f\|_{H^{1/2}(\partial\Omega)}, \quad \text{for } d=3,$$

where

$$\mathcal{E}_1^*(\xi) = \begin{cases} \left(\hat{v}_{1i}^*(\xi) - \xi_i - \dfrac{1}{2\pi}\left(1-\dfrac{\mu_0}{\mu_*}\right)\displaystyle\int_{\partial B}\xi_j' \frac{\partial \hat{v}_{1i}^{\mu}}{\partial\nu}\Big|_-(\xi')\,\mathrm{d}s(\xi')\frac{\xi_j}{|\xi|^2}\right)\partial_i E_0(z) & \text{for } d=2,\\ (\hat{v}_{1i}^*(\xi) - \xi_i)\partial_i E_0(z) & \text{for } d=3, \end{cases}$$

and

$$\mathcal{E}_2^*(\xi) = \begin{cases} \left(\hat{v}_{2ij}^*(\xi) - \dfrac{1}{2}\xi_i\xi_j\right)\partial_{ij}^2 E_0(z) - \dfrac{k^2}{2\pi}\left(1-\dfrac{\varepsilon_*}{\varepsilon_0}\right)|B|E_0(z)\log|\xi| & \text{for } d=2,\\ \left(\hat{v}_{2ij}^*(\xi) - \dfrac{1}{2}\xi_i\xi_j\right)\partial_{ij}^2 E_0(z) & \text{for } d=3. \end{cases}$$

We may follow a variational approach for doing this. We first integrate by parts to see from use of the Helmholtz equation (2.4) for $E_0$ that

$$\int_{z+\alpha B}\operatorname{grad} E_0\cdot\operatorname{grad} v = k^2\int_{z+\alpha B} E_0 v + \int_{\partial(z+\alpha B)}\frac{\partial E_0}{\partial\nu}v,$$

which yields

$$a_\alpha^{\mathrm{TE}}(E_\alpha - E_0, v) = \omega^2 \varepsilon_0 \left( \frac{\varepsilon_*}{\varepsilon_0} - \frac{\mu_0}{\mu_*} \right) \int\limits_{z+\alpha B} E_0 v + \left( \frac{1}{\mu_0} - \frac{1}{\mu_*} \right) \int\limits_{\partial(z+\alpha B)} \frac{\partial E_0}{\partial \nu} v,$$

for any $v \in H_0^1(\Omega)$. We start with estimating the term $\int_{z+\alpha B} E_0 v$. Extend $v$ to the entire $\mathbb{R}^d$ as zero and denote $V$ this extension to find that

$$\begin{aligned}
\left| \int\limits_{z+\alpha B} E_0(x) v(x)\,\mathrm{d}x \right| &\leqslant \left| \int\limits_{z+\alpha B} E_0(x) V(x)\,\mathrm{d}x \right| = \alpha^d \left| \int\limits_B E_0(z+\alpha\xi) V(z+\alpha\xi)\,\mathrm{d}\xi \right| \\
&\leqslant C\alpha^d \left( \int\limits_B \left| V(z+\alpha\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \\
&\leqslant C\alpha^3 \left( \int\limits_{\mathbb{R}^3} \frac{|V(z+\alpha\xi)|^2}{1+|\xi|^2}\,\mathrm{d}\xi \right)^{1/2} \quad (\text{if } d=3) \\
&\leqslant C\alpha^2 \left( \int\limits_{\mathbb{R}^2} \frac{|V(z+\alpha\xi)|^2}{(1+|\xi|^2)\log(2+|\xi|^2)^2}\,\mathrm{d}\xi \right)^{1/2} \quad (\text{if } d=2).
\end{aligned}$$

Since the function $V(z+\alpha\cdot)$ has compact support and as

$$\operatorname{grad} V(z+\alpha\cdot) \in L^2\big(\mathbb{R}^d\big),$$

it follows from the weighted Sobolev compact imbeddings [1] that for $d=3$,

$$\begin{aligned}
\left( \int\limits_{\mathbb{R}^3} \frac{|V(z+\alpha\xi)|^2}{1+|\xi|^2}\,\mathrm{d}\xi \right)^{1/2} &\leqslant C \left( \int\limits_{\mathbb{R}^3} \left| \operatorname{grad} V(z+\alpha\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \\
&\leqslant C\alpha^{-1/2} \left( \int\limits_{\mathbb{R}^3} \left| \operatorname{grad} V(x) \right|^2 \mathrm{d}x \right)^{1/2} \leqslant C\alpha^{-1/2} \|v\|_{H^1(\Omega)},
\end{aligned}$$

while for $d=2$

$$\begin{aligned}
\left( \int\limits_{\mathbb{R}^2} \frac{|V(z+\alpha\xi)|^2}{(1+|\xi|^2)\log(2+|\xi|^2)^2}\,\mathrm{d}\xi \right)^{1/2} &\leqslant C \left( \int\limits_{\mathbb{R}^2} \left| \operatorname{grad} V(z+\alpha\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \\
&\leqslant C \left( \int\limits_{\mathbb{R}^2} \left| \operatorname{grad} V(x) \right|^2 \mathrm{d}x \right)^{1/2} \leqslant C\|v\|_{H^1(\Omega)}.
\end{aligned}$$

A combination of the three last estimates gives:

$$\left|\int_{z+\alpha B} E_0(x)v(x)\,\mathrm{d}x\right| \leqslant C\alpha^{d/2+1}\|v\|_{H^1(\Omega)} \tag{2.60}$$

for any $v \in H_0^1(\Omega)$.

We begin by proving Theorem 2.1 for $d = 3$. We proceed to estimate

$$a_\alpha^{\mathrm{TE}}\left(E_\alpha - E_0 - \alpha\mathcal{E}_1^*\left(\frac{x-z}{\alpha}\right), v\right)$$

where $v$ is any function in $H_0^1(\Omega)$. We first note that $\mathcal{E}_j^*$, $j = 1, 2$, belongs to $H_0^1(\Omega)$ if $d = 3$ and the jump of the normal derivative of $\mathcal{E}_1^*$ across $\partial(z+\alpha B)$ has the special form:

$$\frac{1}{\mu_0}\frac{\partial\mathcal{E}_1^*}{\partial\nu}\bigg|_+ - \frac{1}{\mu_*}\frac{\partial\mathcal{E}_1^*}{\partial\nu}\bigg|_- = -\frac{1}{\alpha}\left(\frac{1}{\mu_0} - \frac{1}{\mu_*}\right)\partial_i E_0(z)\nu_i \quad \text{if } d = 3.$$

Furthermore, the function $\mathcal{E}_1^*$ satisfies

$$\Delta\mathcal{E}_1^* = 0 \quad \text{if } d = 3$$

in $z + \alpha B$ and $\Omega \setminus \overline{z+\alpha B}$. Integration by parts yields

$$\begin{aligned}
&a_\alpha^{\mathrm{TE}}\left(E_\alpha - E_0 - \alpha\mathcal{E}_1^*, v\right)\\
&\quad = \omega^2\varepsilon_0\left(\frac{\varepsilon_*}{\varepsilon_0} - \frac{\mu_0}{\mu_*}\right)\int_{z+\alpha B} E_0 v + \left(\frac{1}{\mu_0} - \frac{1}{\mu_*}\right)\int_{\partial(z+\alpha B)}\left(\frac{\partial E_0}{\partial\nu} - \partial_i E_0(z)\nu_i\right)v\\
&\qquad + \alpha\omega^2\int_\Omega \varepsilon_\alpha\mathcal{E}_1^* v
\end{aligned}$$

for any $v \in H_0^1(\Omega)$. Since

$$\int_{\partial(z+\alpha B)}\left(\frac{\partial E_0}{\partial\nu} - \partial_i E_0(z)\nu_i\right)v = -k^2\int_{z+\alpha B} E_0 v + \int_{z+\alpha B}\left(\operatorname{grad} E_0(x) - \partial_i E_0(z)\mathbf{u}_i\right)\cdot\operatorname{grad} v,$$

where $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ is the standard basis of $\mathbb{R}^d$, we may rewrite

$$a_\alpha^{\mathrm{TE}}\left(E_\alpha - E_0 - \alpha\mathcal{E}_1^*, v\right)$$

as

$$
\begin{aligned}
a_\alpha^{\mathrm{TE}}\big(E_\alpha - E_0 - \alpha\mathcal{E}_1^*, v\big) = \omega^2(\varepsilon_* - \varepsilon_0) \int\limits_{z+\alpha B} E_0(x) v(x)\,\mathrm{d}x \\
+ \left(\frac{1}{\mu_0} - \frac{1}{\mu_*}\right) \int\limits_{z+\alpha B} \big(\operatorname{grad} E_0(x) - \partial_i E_0(z)\mathbf{u}_i\big)\cdot \operatorname{grad} v(x)\,\mathrm{d}x \\
+ \alpha\omega^2 \int\limits_\Omega \varepsilon_\alpha \mathcal{E}_1^*\left(\frac{x-z}{\alpha}\right) v(x)\,\mathrm{d}x, \qquad (2.61)
\end{aligned}
$$

for any $v \in H_0^1(\Omega)$. We shall estimate the terms in the above right-hand side one by one to obtain the following:

**Lemma 2.7.** *There exists a constant $C$, independent of $\alpha$ and $f$, such that*

$$
\sup_{v\in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)}=1} \left| a_\alpha^{\mathrm{TE}}(E_\alpha - E_0 - \alpha\mathcal{E}_1^*, v)\right| \leqslant C\alpha^{5/2} \quad \textit{if } d = 3. \qquad (2.62)
$$

**Proof.** We first start with the last term of (2.61). Using the decay of $\hat{v}_{1i}^{\mu} - \xi_i$ at infinity we get for any positive $p$,

$$
\begin{aligned}
\left(\int\limits_\Omega \left|\mathcal{E}_1^*\left(\frac{x-z}{\alpha}\right)\right|^p \mathrm{d}x\right)^{1/p} &= \alpha^{d/p}\left(\int\limits_{(\Omega-z)/\alpha} \left|\mathcal{E}_1^*(\xi)\right|^p \mathrm{d}\xi\right)^{1/p} \\
&\leqslant \alpha^{d/p}\left(\int\limits_{\mathbb{R}^d} \left|\mathcal{E}_1^*(\xi)\right|^p \mathrm{d}\xi\right)^{1/p}.
\end{aligned}
$$

Therefore

$$
\left|\int\limits_\Omega \varepsilon_\alpha \mathcal{E}_1^* v\right| \leqslant C\left(\int\limits_\Omega \left|\mathcal{E}_1^*\left(\frac{x-z}{\alpha}\right)\right|^p \mathrm{d}x\right)^{1/p} \|v\|_{L^{p'}(\Omega)} \leqslant C\alpha^{d/p}\|v\|_{L^{p'}(\Omega)} \qquad (2.63)
$$

for any positive $p$ and $p'$, such that $1/p + 1/p' = 1$. Since $\Omega$ is a bounded, $C^2$-domain of $\mathbb{R}^d$, we know from the Sobolev Imbedding Theorem [1] that the imbedding $H^1(\Omega) \hookrightarrow L^{p'}(\Omega)$ is continuous for any $2 \leqslant p' \leqslant 6$ if $d = 3$ (and for any $1 < p' < +\infty$ if $d = 2$). We may choose $p' = 6$ if $d = 3$, in which case

$$
\left|\int\limits_\Omega \varepsilon_\alpha \mathcal{E}_1^* v\right| \leqslant C\alpha^{5/2}\|v\|_{H^1(\Omega)} \quad \text{if } d = 3. \qquad (2.64)
$$

These are the desired estimates for the last term of (2.61). The second term in (2.61) is very simple to estimate. Using the interior regularity of the field $E_0$, we obtain:

$$\begin{aligned}&\left|\int\limits_{z+\alpha B}\big(\operatorname{grad}E_0(x)-\partial_i E_0(z)\mathbf{u}_i\big)\cdot\operatorname{grad}v\right|\\&\quad\leqslant \alpha^{d/2}\big\|\operatorname{grad}E_0(x)-\partial_i E_0(z)\mathbf{u}_i\big\|_{L^\infty(z+\alpha B)}\|v\|_{H^1(\Omega)}\\&\quad\leqslant C\alpha^{d/2+1}\|v\|_{H^1(\Omega)}.\end{aligned}\tag{2.65}$$

A combination of the estimates (2.60), (2.64), and (2.65) yields the desired result. □

We now proceed further in the construction of the asymptotic expansion of $E_\alpha$ by proving the following:

**Lemma 2.8.** *There exists a constant $C$, independent of $\alpha$ and $f$, such that*

$$\sup_{v\in H_0^1(\Omega),\,\|v\|_{H^1(\Omega)}=1}\left|a_\alpha^{\mathrm{TE}}\big(E_\alpha-E_0-\alpha\mathcal{E}_1^*-\alpha^2\mathcal{E}_2^*,v\big)\right|\leqslant C\alpha^3\quad \textit{if } d=3.\tag{2.66}$$

**Proof.** First we note that

$$\begin{aligned}\operatorname{div}\frac{1}{\mu_\alpha}\operatorname{grad}\mathcal{E}_2^*\left(\frac{x-z}{\alpha}\right)&=\left(\frac{1}{\mu_0}\frac{\partial\mathcal{E}_2^*}{\partial\nu}\bigg|_+-\frac{1}{\mu_*}\frac{\partial\mathcal{E}_2^*}{\partial\nu}\bigg|_-\right)\delta_{\partial(z+\alpha B)}\\&\quad-\frac{\omega^2}{\alpha^2}\left(\varepsilon_*-\frac{\varepsilon_0\mu_0}{\mu_*}\right)E_0(z)\chi(z+\alpha B),\end{aligned}\tag{2.67}$$

where $\chi(z+\alpha B)$ is the characteristic function of the domain $z+\alpha B$ and $\delta_{\partial(z+\alpha B)}$ is the surface Dirac measure at $\partial(z+\alpha B)$.

Using the fact that

$$\frac{1}{\mu_0}\frac{\partial\mathcal{E}_2^*}{\partial\nu}\bigg|_+-\frac{1}{\mu_*}\frac{\partial\mathcal{E}_2^*}{\partial\nu}\bigg|_-=-\frac{1}{2\alpha}\left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)\partial_{ij}E_0(z)(\nu_i\xi_j+\nu_j\xi_i)\quad\text{on }\partial B,$$

we readily obtain from (2.67) that

$$\begin{aligned}&a_\alpha^{\mathrm{TE}}\big(E_\alpha-E_0-\alpha\mathcal{E}_1^*-\alpha^2\mathcal{E}_2^*,v\big)\\&\quad=\omega^2(\varepsilon_*-\varepsilon_0)\int\limits_{z+\alpha B}\big(E_0(x)-E_0(z)\big)v(x)\,\mathrm{d}x\\&\qquad+\left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)\int\limits_{z+\alpha B}\bigg(\operatorname{grad}E_0(x)-\partial_iE_0(z)\mathbf{u}_i\\&\qquad\qquad-\frac{1}{2}(x-z)_j(x-z)_i\partial_{ij}E_0(z)\mathbf{u}_i\bigg)\cdot\operatorname{grad}v(x)\,\mathrm{d}x\\&\qquad+\alpha\omega^2\int\limits_\Omega\varepsilon_\alpha\big(\mathcal{E}_1^*+\alpha\mathcal{E}_2^*\big)\left(\frac{x-z}{\alpha}\right)v(x)\,\mathrm{d}x,\end{aligned}\tag{2.68}$$

for any $v \in H_0^1(\Omega)$. Finally, we note that the same arguments used in (2.64) can be used here to establish that

$$\left| \int_{\Omega} \varepsilon_\alpha \mathcal{E}_2^* v \right| \leqslant C\alpha^{5/2} \|v\|_{H^1(\Omega)} \quad \text{if } d = 3. \tag{2.69}$$

Combining now (2.64) and (2.69), we find from use of interior regularity of $E_0$ once more to estimate the term

$$\int_{z+\alpha B} \left( \operatorname{grad} E_0(x) - \partial_i E_0(z)\mathbf{u}_i - \frac{1}{2}(x-z)_j(x-z)_i \partial_{ij} E_0(z)\mathbf{u}_i \right) \cdot \operatorname{grad} v(x)\,\mathrm{d}x$$

that estimates (2.66) hold. $\square$

For $d = 2$ the following lemma holds.

**Lemma 2.9.** *There exists a constant $C$, independent of $\alpha$ and $f$, such that*

$$\begin{aligned} &\sup_{v \in H_0^1(\Omega),\, \|v\|_{H^1(\Omega)}=1} \left| a_\alpha^{\mathrm{TE}}\big(E_\alpha - E_0 - \alpha^2 E_1 - \alpha \mathcal{E}_1^* - \alpha^2 \mathcal{E}_2^*, v\big) \right| \\ &\quad \leqslant C\alpha^{3-\delta} \quad \text{if } d = 2, \end{aligned} \tag{2.70}$$

*for any fixed $\delta > 0$.*

**Proof** (*outline*). We first note that although each of the terms $E_1(x), \mathcal{E}_1^*((x-z)/\alpha)$, and $\mathcal{E}_2^*((x-z)/\alpha)$ is singular as $x \to z$ the sum $\mathcal{E}_1^* + \alpha E_1 + \alpha \mathcal{E}_2^*$ belongs to $H_0^1(\Omega)$. The expression of $a_\alpha^{\mathrm{TE}}(E_\alpha - E_0 - \alpha^2 E_1 - \alpha \mathcal{E}_1^* - \alpha^2 \mathcal{E}_2^*, v)$ for $v \in H_0^1(\Omega)$ is slightly different from (2.68). In the case where $d = 2$ the right-hand side of (2.68) should be replaced by:

$$\begin{aligned} &\omega^2(\varepsilon_* - \varepsilon_0) \int_{z+\alpha B} \big(E_0(x) - E_0(z)\big) v(x)\,\mathrm{d}x \\ &\quad + \left(\frac{1}{\mu_0} - \frac{1}{\mu_*}\right) \int_{z+\alpha B} \Bigg( \operatorname{grad} E_0(x) - \partial_i E_0(z)\mathbf{u}_i \\ &\qquad\qquad - \frac{1}{2}(x-z)_j(x-z)_i \partial_{ij} E_0(z)\mathbf{u}_i \Bigg) \cdot \operatorname{grad} v(x)\,\mathrm{d}x \\ &\quad + \alpha\omega^2 \int_{z+\alpha B} \varepsilon_\alpha \left[ \big(\mathcal{E}_1^* + \alpha \mathcal{E}_2^*\big)\left(\frac{x-z}{\alpha}\right) + \alpha E_1(x) \right] v(x)\,\mathrm{d}x \\ &\quad + \alpha\omega^2 \int_{\Omega \setminus \overline{z+\alpha B}} \varepsilon_\alpha \left[ \big(\mathcal{E}_1^* + \alpha \mathcal{E}_2^*\big)\left(\frac{x-z}{\alpha}\right) \right] v(x)\,\mathrm{d}x. \end{aligned}$$

It is not difficult to prove that the third term in this expression can be estimated by $C\alpha^3$ because of the smoothness of $[(\mathcal{E}_1^* + \alpha\mathcal{E}_2^*)((x-z)/\alpha) + \alpha E_1(x)]$. To estimate the last term we use the same (scaling) argument as for establishing (2.63) where in the present case $1/p' = 1 - \delta$ ($\delta > 0$). $\square$

We are now ready to proceed with the proof of Theorem 2.1. Using Lemma 2.1 we obtain from estimates (2.66) and (2.70) that

$$\left\| E_\alpha - E_0 - \alpha\mathcal{E}_1^* - \alpha^2\mathcal{E}_2^* \right\|_{H^1(\Omega)} \leqslant C\alpha^3 \quad \text{if } d = 3, \tag{2.71}$$

$$\left\| E_\alpha - E_0 - \alpha\mathcal{E}_1^* - \alpha^2 E_1 - \alpha^2\mathcal{E}_2^* \right\|_{H^1(\Omega)} \leqslant C\alpha^{3-\delta} \quad \text{if } d = 2, \tag{2.72}$$

for any fixed $\delta > 0$. Theorem 2.1 follows from Lemma 2.6. The proof of Theorem 2.1 is now complete. $\square$

Similar analysis may be carried out to justify the asymptotic expansions of the magnetic field $H_\alpha$ in the TM case. Here, we only emphasize the differences needed to readily prove that similar expansions to those stated in Theorem 2.1 hold for the magnetic field $H_\alpha$ which is defined as the unique solution to the Helmholtz equation (2.13) with the Neumann boundary condition:

$$\frac{1}{\varepsilon_\alpha}\frac{\partial H_\alpha}{\partial \nu} = g \in H^{-1/2}(\partial\Omega) \quad \text{on } \partial\Omega.$$

Introduce in this case:

$$\begin{aligned}
H_1(x) &= \left(1 - \frac{\varepsilon_0}{\varepsilon_*}\right)\partial_i H_0(z)\left(\int_B \operatorname{grad}\hat{v}_{1i}^{\varepsilon}(\xi)\,\mathrm{d}\xi\right)\cdot \operatorname{grad} G_N(x,z) \\
&\quad + k^2\left(1 - \frac{\mu_*}{\mu_0}\right)|B| H_0(z) G_N(x,z) \quad \text{for } x \neq z, \\
H_2(x) &= \left(1 - \frac{\varepsilon_0}{\varepsilon_*}\right)\left[\partial_{ij}^2 H_0(z)\left(\int_B \operatorname{grad}\hat{v}_{2ij}^{\varepsilon,\mu}(\xi)\,\mathrm{d}\xi\right)\cdot \operatorname{grad} G_N(x,z)\right. \\
&\qquad \left. + \partial_i H_0(z)\left(\int_B \xi_j\,\partial_k \hat{v}_{1i}^{\varepsilon}(\xi)\,\mathrm{d}\xi\right)\partial_{jk}^2 G_N(x,z)\right] \\
&\quad + k^2\left(\frac{\mu_*}{\mu_0} - 1\right)\left[\partial_i H_0(z)\left(\int_B \hat{v}_{1i}^{\varepsilon}(\xi)\,\mathrm{d}\xi\right) G_N(x,z)\right. \\
&\qquad \left. + H_0(z)\left(\int_B \xi_j\,\mathrm{d}\xi\right)\partial_j G_N(x,z)\right] \quad \text{for } x \neq z,
\end{aligned}$$

and the auxiliary fields:

$$\mathcal{H}_1(\xi) = \begin{cases} \left(\hat{v}_{1i}^{\varepsilon}(\xi) - \xi_i - \frac{1}{2\pi}\left(1 - \frac{\varepsilon_0}{\varepsilon_*}\right)\int_{\partial B} \xi_j' \frac{\partial \hat{v}_{1i}^{\varepsilon}}{\partial \nu}\Big|_{-}(\xi')\,\mathrm{d}s(\xi')\frac{\xi_j}{|\xi|^2}\right)\partial_i H_0(z) & \text{for } d = 2, \\ \left(\hat{v}_{1i}^{\varepsilon}(\xi) - \xi_i\right)\partial_i H_0(z) & \text{for } d = 3, \end{cases}$$

and

$$\mathcal{H}_2(\xi) = \begin{cases} \left(\hat{v}_{2ij}^{\varepsilon,\mu}(\xi) - \frac{1}{2}\xi_i\xi_j\right)\partial_{ij}^2 H_0(z) - \frac{k^2}{2\pi}\left(1 - \frac{\mu_*}{\mu_0}\right)|B| H_0(z)\log|\xi| & \text{for } d = 2, \\ \left(\hat{v}_{2ij}^{\varepsilon,\mu}(\xi) - \frac{1}{2}\xi_i\xi_j\right)\partial_{ij}^2 H_0(z) & \text{for } d = 3. \end{cases}$$

The following estimates for the TM case hold.

**Theorem 2.2.** *Suppose* (2.1) *and* (2.15) *are satisfied. There exists a constant $C$, independent of $\alpha$ and $g$, such that*:

- *For $d = 2$*

$$\left\| H_\alpha(x) - H_0(x) - \alpha\mathcal{H}_1\left(\frac{x-z}{\alpha}\right) - \alpha^2 H_1(x) - \alpha^2\mathcal{H}_2\left(\frac{x-z}{\alpha}\right)\right\|_{H^1(\Omega)} \leqslant C\alpha^{3-\delta}\|g\|_{H^{-1/2}(\partial\Omega)},$$

  *for any fixed $\delta > 0$.*
- *For $d = 3$*

$$\left\| H_\alpha(x) - H_0(x) - \alpha\mathcal{H}_1\left(\frac{x-z}{\alpha}\right) - \alpha^2\mathcal{H}_2\left(\frac{x-z}{\alpha}\right)\right\|_{H^1(\Omega)} \leqslant C\alpha^3\|g\|_{H^{-1/2}(\partial\Omega)}.$$

-

$$\left\| H_\alpha(x) - H_0(x) - \alpha^d H_1(x) - \alpha^{d+1} H_2(x)\right\|_{L^\infty(\partial\Omega)} \leqslant C \begin{cases} \alpha^{4-\delta}\|g\|_{H^{-1/2}(\partial\Omega)} & \textit{if } d = 2, \\ \alpha^5\|g\|_{H^{-1/2}(\partial\Omega)} & \textit{if } d = 3. \end{cases}$$

*The constant $C$ depends on the domains $\{B_j\}_{j=1}^m$, $\Omega$, the constants $\{\mu_j, \varepsilon_j\}_{j=0}^m$, and $c_0$, but is otherwise independent of the points $\{z_j\}_{j=1}^m$.*

**Proof.** Following our analysis of the TE case, let us compute for $d = 3$

$$a_\alpha^{\mathrm{TM}}\left(H_\alpha - H_0 - \alpha\mathcal{H}_1 - \alpha^2\mathcal{H}_2, v\right)$$

for all $v \in H^1(\Omega)$. Theorem 2.2 will then be obtained by applying Lemma 2.2.

We have:

$$\begin{aligned}
&a_\alpha^{\mathrm{TM}}\big(H_\alpha - H_0 - \alpha\mathcal{H}_1 - \alpha^2\mathcal{H}_2, v\big)\\
&\quad = \omega^2(\mu_* - \mu_0)\int\limits_{z+\alpha B}\big(H_0(x) - H_0(z)\big)v(x)\,\mathrm{d}x\\
&\qquad + \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_*}\right)\int\limits_{z+\alpha B}\bigg(\operatorname{grad} H_0(x) - \partial_i H_0(z)\mathbf{u}_i\\
&\qquad\qquad - \frac{1}{2}(x-z)_i(x-z)_j\partial_{ij}H_0(z)\mathbf{u}_i\bigg)\cdot\operatorname{grad} v\\
&\qquad - \alpha\int\limits_{\partial\Omega}\left(\frac{\partial\mathcal{H}_1}{\partial\nu} + \alpha\frac{\partial\mathcal{H}_2}{\partial\nu}\right)v + \alpha\omega^2\int\limits_{\Omega}\mu_\alpha(\mathcal{H}_1 + \mathcal{H}_2)v,
\end{aligned}\tag{2.73}$$

for any $v \in H^1(\Omega)$.

In order to estimate the term $\int_{z+\alpha B}(H_0(x) - H_0(z))v(x)\,\mathrm{d}x$ we need to extend $v$ from $\Omega$ to all of $\mathbb{R}^d$. In that regard let $\mathcal{B}$ be a sufficiently large open ball that $\overline{\Omega} \subset \mathcal{B}$. It is well known that there exists $\tilde{v} \in H^1(\mathcal{B} \setminus \overline{\Omega})$ such that $\tilde{v} = v$ on $\partial\Omega$, $\tilde{v} = 0$ on $\partial\mathcal{B}$, and

$$\|\tilde{v}\|_{H^1(\mathcal{B}\setminus\overline{\Omega})} \leqslant C\|v\|_{H^{1/2}(\partial\Omega)} \leqslant C\|v\|_{H^1(\Omega)}.$$

For the last estimate we use the trace inequality. Now define

$$V(x) = \begin{cases} v(x), & x \in \Omega,\\ \tilde{v}(x), & x \in \mathcal{B}\setminus\overline{\Omega},\\ 0, & x \in \mathbb{R}^d \setminus \overline{\mathcal{B}}.\end{cases}$$

Introducing this $V$ we may now write

$$\begin{aligned}
\left|\int\limits_{z+\alpha B} H_0(x)v(x)\,\mathrm{d}x\right| &\leqslant \left|\int\limits_{z+\alpha B} H_0(x)V(x)\,\mathrm{d}x\right| = \alpha^d\left|\int\limits_{B} H_0(z+\alpha\xi)V(z+\alpha\xi)\,\mathrm{d}\xi\right|\\
&\leqslant C\alpha^d\left(\int\limits_B \big|V(z+\alpha\xi)\big|^2\,\mathrm{d}\xi\right)^{1/2}.
\end{aligned}$$

Since the function $V(z+\alpha\cdot)$ has compact support and as $\operatorname{grad} V(z+\alpha\cdot) \in L^2(\mathbb{R}^d)$ it can be seen from the weighted Sobolev compact imbeddings and by use of interior regularity of $H_0$, exactly as it has been done for the TE case, that the following estimate holds:

$$\left|\int\limits_{z+\alpha B}\big(H_0(x) - H_0(z)\big)v(x)\,\mathrm{d}x\right| \leqslant C\alpha^{d/2+2}\|v\|_{H^1(\Omega)}$$

for any $v \in H^1(\Omega)$.

The terms $\int_{\partial\Omega} \frac{\partial \mathcal{H}_1}{\partial \nu} v$ and $\int_{\partial\Omega} \frac{\partial \mathcal{H}_2}{\partial \nu} v$ in (2.73) are very simple to estimate. Using the $H^1$ extension $V$ of $v$ to the entire $\mathbb{R}^d$ as it has been constructed earlier and the fact that the function $\mathcal{H}_1$ is harmonic inside $\mathcal{B} \setminus \overline{\Omega}$, we obtain:

$$\int_{\partial\Omega} \frac{\partial \mathcal{H}_1}{\partial \nu} v = -\int_{\mathcal{B}\setminus\overline{\Omega}} \operatorname{grad} \mathcal{H}_1 \cdot \operatorname{grad} V.$$

Using the decay of $\hat{v}_{1i}^{\varepsilon} - \xi_i$ at infinity we get:

$$\begin{aligned}
\left| \int_{\mathcal{B}\setminus\overline{\Omega}} \operatorname{grad} \mathcal{H}_1 \cdot \operatorname{grad} V \right| &\leqslant C \left( \int_{\mathcal{B}\setminus\overline{\Omega}} (\operatorname{grad} \mathcal{H}_1)^2 \right)^{1/2} \|V\|_{H^1(\mathcal{B}\setminus\overline{\Omega})} \\
&\leqslant C\alpha^{d/2-1} \left( \int_{(\mathcal{B}-z)/\alpha \setminus (\overline{\Omega - z})/\alpha} (\operatorname{grad}_\xi \mathcal{H}_1)^2 \right)^{1/2} \|v\|_{H^1(\Omega)} \\
&\leqslant C\alpha^{d/2-1} \left( \int_{r_1/\alpha}^{r_2/\alpha} \frac{r^{d-1}}{r^{2d}} \,\mathrm{d}r \right)^{1/2} \|v\|_{H^1(\Omega)} \leqslant C\alpha^{d-1} \|v\|_{H^1(\Omega)}.
\end{aligned}$$

In a completely similar manner, we obtain from the decay of $\hat{v}_{2ij}^{\varepsilon,\mu} - \frac{1}{2}\xi_i\xi_j$ at infinity that

$$\left| \int_{\mathcal{B}\setminus\overline{\Omega}} \operatorname{grad} \mathcal{H}_2 \cdot \operatorname{grad} V \right| \leqslant C\alpha^{d-1} \|v\|_{H^1(\Omega)}.$$

Now, application of Lemma 2.2 yields the desired estimates. $\square$

Theorems 2.1 and 2.2 provide a proof of the asymptotic expansions of the fields $E_\alpha$ and $H_\alpha$, for the case of one inhomogeneity. The proof for any fixed number of well separated inhomogeneities follows essentially by iteration of the arguments just presented. The statements of Theorems 2.1 and 2.2 become in this case:

$$\begin{aligned}
&\left\| E_\alpha(x) - E_0(x) - \alpha \mathcal{E}_{1j}\left( \frac{x - z_j}{\alpha} \right) - \alpha^2 E_{1j}(x) - \alpha^2 \mathcal{E}_{2j}\left( \frac{x - z_j}{\alpha} \right) \right\|_{H^1(\Omega)} \\
&\quad \leqslant C\alpha^{3-\delta} \|f\|_{H^{1/2}(\partial\Omega)}, \\
&\left\| H_\alpha(x) - H_0(x) - \alpha \mathcal{H}_{1j}\left( \frac{x - z_j}{\alpha} \right) - \alpha^2 H_{1j}(x) - \alpha^2 \mathcal{H}_{2j}\left( \frac{x - z_j}{\alpha} \right) \right\|_{H^1(\Omega)} \\
&\quad \leqslant C\alpha^{3-\delta} \|g\|_{H^{-1/2}(\partial\Omega)}, \quad d = 2,
\end{aligned}$$

$$\left\| E_\alpha(x) - E_0(x) - \alpha \mathcal{E}_{1j}\left(\frac{x-z_j}{\alpha}\right) - \alpha^2 \mathcal{E}_{2j}\left(\frac{x-z_j}{\alpha}\right) \right\|_{H^1(\Omega)} \leqslant C\alpha^3 \|f\|_{H^{1/2}(\partial\Omega)},$$

$$\left\| H_\alpha(x) - H_0(x) - \alpha \mathcal{H}_{1j}\left(\frac{x-z_j}{\alpha}\right) - \alpha^2 \mathcal{H}_{2j}\left(\frac{x-z_j}{\alpha}\right) \right\|_{H^1(\Omega)}$$
$$\leqslant C\alpha^3 \|g\|_{H^{-1/2}(\partial\Omega)}, \quad d=3,$$

$$\left\| \frac{\partial E_\alpha(x)}{\partial \nu(x)} - \frac{\partial E_0(x)}{\partial \nu(x)} - \alpha^d \frac{\partial E_{1j}(x)}{\partial \nu(x)} - \alpha^{d+1} \frac{\partial E_{2j}(x)}{\partial \nu(x)} \right\|_{L^\infty(\partial\Omega)}$$
$$\leqslant C \begin{cases} \alpha^{4-\delta} \|f\|_{H^{1/2}(\partial\Omega)} & \text{if } d=2, \\ \alpha^5 \|f\|_{H^{1/2}(\partial\Omega)} & \text{if } d=3, \end{cases}$$

$$\left\| H_\alpha(x) - H_0(x) - \alpha^d H_{1j}(x) - \alpha^{d+1} H_{2j}(x) \right\|_{L^\infty(\partial\Omega)} \leqslant C \begin{cases} \alpha^{4-\delta} \|g\|_{H^{-1/2}(\partial\Omega)} & \text{if } d=2, \\ \alpha^5 \|g\|_{H^{-1/2}(\partial\Omega)} & \text{if } d=3, \end{cases}$$

where the constant $C$ depends on the domains $\{B_j\}_{j=1}^m$, $\Omega$, the constants $\{\mu_j, \varepsilon_j\}_{j=0}^m$, and $c_0$, but is otherwise independent of the points $\{z_j\}_{j=1}^m$. Here, the functions $(\mathcal{E}_{ij}, \mathcal{H}_{ij})$ are exactly the functions $(\mathcal{E}_i, \mathcal{H}_i)$ that are calculated for the domain $B = B_j$ and the electromagnetic parameters $(\varepsilon_*, \mu_*) = (\varepsilon_j, \mu_j)$. The same definitions are adopted for $E_{1j}$, $H_{1j}$, $E_{2j}$, and $H_{2j}$.

## 3. Properties of the generalized polarization tensors

In this section we analyze in some detail the properties of the generalized polarization tensors $M^{1,1}$, $M^{1,2}$, and $M^{2,1}$, introduced earlier. These generalized polarization tensors seem to carry out significant information on the small dielectric inhomogeneities which is yet to be investigated. Although this part focuses on the generalized polarization tensors that appear in higher-order asymptotics of TE fields, all the results are easily carried over to the case of TM fields but just exchanging the roles of $\varepsilon$ and $\mu$. In Section 3.3. we will study the limits of the generalized polarization tensors $M^{1,1}$, $M^{1,2}$, and $M^{2,1}$ as $\mu_*/\mu_0$ tends towards zero or infinity. To end up with a correct physical interpretation of these asymptotic results we should rather interchange the roles of $\varepsilon$ and $\mu$.

We recall that

$$M_{ij}^{1,1} = \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_j \, \mathrm{d}s(\xi),$$

$$M_{ijk}^{1,2} = \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_j\xi_k \, \mathrm{d}s(\xi), \qquad M_{ijk}^{2,1} = \int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_{-}(\xi)\xi_k \, \mathrm{d}s(\xi).$$

The generalized polarization tensors $M^{1,1}$, $M^{1,2}$, and $M^{2,1}$ can be calculated explicitly in the case $B$, a ball or a two-dimensional ellipse [12,15].

### 3.1. Technical identities

The following lemma will be useful later.

**Lemma 3.1.** *The following identities hold.*

$$\text{(a)} \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_j \,\mathrm{d}s(\xi) = \int_{\partial B} \frac{\partial \hat{v}_{1j}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_i \,\mathrm{d}s(\xi).$$

$$\text{(b)} \left(\frac{1}{\mu_*}+\frac{1}{\mu_0}\right)\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_j \,\mathrm{d}s(\xi) = \frac{1}{\mu_0}|B|\delta_{ij} + \frac{1}{\mu_*}\int_B \operatorname{grad}\hat{v}_{1i}^{\mu}(\xi)\cdot \operatorname{grad}\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}\xi$$
$$+ \frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot \operatorname{grad}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}\xi.$$

$$\text{(c)} \ \frac{1}{2}\left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_j\xi_k \,\mathrm{d}s(\xi)$$
$$= \delta_{jk}\frac{1}{\mu_0}\left(\frac{\varepsilon_*}{\varepsilon_0}-1\right)\int_B \big(\hat{v}_{1i}^{\mu}-\xi_i\big)\,\mathrm{d}\xi + \frac{1}{2}\left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)\int_B (\xi_j\delta_{ki}+\xi_k\delta_{ij})\,\mathrm{d}\xi$$
$$+ \frac{1}{\mu_*}\int_B \operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot \operatorname{grad}\left(\hat{v}_{2jk}^{\mu,\varepsilon}-\frac{1}{2}\xi_j\xi_k\right)\mathrm{d}\xi$$
$$+ \frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot \operatorname{grad}\left(\hat{v}_{2jk}^{\mu,\varepsilon}-\frac{1}{2}\xi_j\xi_k\right)\mathrm{d}\xi.$$

$$\text{(d)} \left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)\int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_{-}(\xi)\xi_k \,\mathrm{d}s(\xi)$$
$$= \delta_{ij}\left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)\frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}\int_B \xi_k\,\mathrm{d}\xi + \frac{1}{2}\left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)\int_B (\xi_i\delta_{jk}+\xi_j\delta_{ik})\,\mathrm{d}\xi$$
$$+ \frac{1}{\mu_*}\int_B \operatorname{grad}\big(\hat{v}_{1k}^{\mu}-\xi_k\big)\cdot \operatorname{grad}\left(\hat{v}_{2ij}^{\mu,\varepsilon}-\frac{1}{2}\xi_i\xi_j\right)\mathrm{d}\xi$$
$$+ \frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1k}^{\mu}-\xi_k\big)\cdot \operatorname{grad}\left(\hat{v}_{2ij}^{\mu,\varepsilon}-\frac{1}{2}\xi_i\xi_j\right)\mathrm{d}\xi.$$

**Proof.** We begin by showing that (a) holds. Using the jump condition for $\partial\hat{v}_{1i}^{\mu}/\partial\nu$ on $\partial B$ and integrating by parts we obtain:

$$\begin{aligned}
\left(\frac{1}{\mu_*}-\frac{1}{\mu_0}\right)\int_{\partial B}\frac{\partial}{\partial\nu}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\big|_-\xi_j\,\mathrm{d}s(\xi) &= \int_{\partial B}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\left(\left(\frac{1}{\mu_*}-\frac{1}{\mu_0}\right)\nu_j\right)\mathrm{d}s(\xi)\\
&= \int_{\partial B}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\left[\frac{1}{\mu_0}\frac{\partial}{\partial\nu}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\big|_+ - \frac{1}{\mu_*}\frac{\partial}{\partial\nu}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\big|_-\right]\mathrm{d}s(\xi)\\
&= \int_{\partial B}\left[\frac{1}{\mu_0}\frac{\partial}{\partial\nu}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\big|_+ - \frac{1}{\mu_*}\frac{\partial}{\partial\nu}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\big|_-\right]\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}s(\xi)\\
&= \left(\frac{1}{\mu_*}-\frac{1}{\mu_0}\right)\int_{\partial B}\nu_i\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}s(\xi)\\
&= \left(\frac{1}{\mu_*}-\frac{1}{\mu_0}\right)\int_{\partial B}\frac{\partial}{\partial\nu}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\big|_-(\xi)\xi_i\,\mathrm{d}s(\xi)
\end{aligned}$$

and therefore, since $\int_{\partial B}\nu_j\xi_i\,\mathrm{d}s(\xi)=\int_{\partial B}\nu_i\xi_j\,\mathrm{d}s(\xi)$, identity (a) holds.

We now prove that (b) holds. Integrating by parts we obtain:

$$\begin{aligned}
\frac{1}{\mu_*}\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_-(\xi)\xi_j\,\mathrm{d}s(\xi) &= \frac{1}{\mu_*}\int_B \operatorname{grad}\hat{v}_{1i}^{\mu}(\xi)\cdot\operatorname{grad}\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}\xi\\
&\quad -\frac{1}{\mu_*}\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_-(\xi)\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}s(\xi).
\end{aligned}$$

The decay of $\hat{v}_{1i}^{\mu}-\xi_i$ implies that the integral

$$\int_{\mathbb{R}^d\setminus\overline{B}}\operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot\operatorname{grad}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}\xi$$

may also be integrated by parts. We have:

$$\frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}}\operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot\operatorname{grad}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}\xi = -\frac{1}{\mu_0}\int_{\partial B}\frac{\partial}{\partial\nu}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\big|_+\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}s(\xi).$$

Using the jump condition for $\partial\hat{v}_{1i}^{\mu}/\partial\nu$ on $\partial B$ once more we get:

$$
\begin{aligned}
&\frac{1}{\mu_*}\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_j \,\mathrm{d}s(\xi) \\
&\quad= \frac{1}{\mu_*}\int_{B} \operatorname{grad}\hat{v}_{1i}^{\mu}(\xi)\cdot \operatorname{grad}\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}\xi + \frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot\operatorname{grad}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}\xi \\
&\qquad+ \frac{1}{\mu_0}\int_{\partial B}\frac{\partial}{\partial\nu}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\big|_{+}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}s(\xi) - \frac{1}{\mu_*}\int_{\partial B}\frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}s(\xi) \\
&\quad= \frac{1}{\mu_*}\int_{B} \operatorname{grad}\hat{v}_{1i}^{\mu}(\xi)\cdot \operatorname{grad}\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}\xi + \frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot\operatorname{grad}\big(\hat{v}_{1j}^{\mu}-\xi_j\big)\,\mathrm{d}\xi \\
&\qquad+ \frac{1}{\mu_0}\int_{\partial B}\nu_i\xi_j\,\mathrm{d}s(\xi) - \frac{1}{\mu_0}\int_{\partial B}\nu_i\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}s(\xi).
\end{aligned}
\tag{3.1}
$$

The last integral on the right-hand side of (3.1) may immediately be integrated by parts to become

$$
-\frac{1}{\mu_0}\int_{\partial B}\frac{\partial \hat{v}_{1j}^{\mu}}{\partial\nu}\bigg|_{-}(\xi)\xi_i\,\mathrm{d}s(\xi).
$$

Noting that $\int_{\partial B}\nu_i\xi_j = |B|\delta_{ij}$ it immediately follows by (a) that (b) holds.

To prove (c) we use (2.35) and (2.42) to compute:

$$
\begin{aligned}
&\int_{B}\operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot\operatorname{grad}\bigg(\hat{v}_{2jk}^{\mu,\varepsilon}-\frac{1}{2}\xi_j\xi_k\bigg)\,\mathrm{d}\xi + \bigg(\frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}-1\bigg)\delta_{jk}\int_{B}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\,\mathrm{d}\xi \\
&\quad= \int_{\partial B}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\frac{\partial}{\partial\nu}\bigg(\hat{v}_{2jk}^{\mu,\varepsilon}-\frac{1}{2}\xi_j\xi_k\bigg)\bigg|_{-}\,\mathrm{d}s(\xi).
\end{aligned}
$$

We also calculate:

$$
\begin{aligned}
&\int_{\mathbb{R}^d\setminus\overline{B}}\operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot\operatorname{grad}\bigg(\hat{v}_{2jk}^{\mu,\varepsilon}-\frac{1}{2}\xi_j\xi_k\bigg)\,\mathrm{d}\xi \\
&\quad= -\int_{\partial B}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\frac{\partial}{\partial\nu}\bigg(\hat{v}_{2jk}^{\mu,\varepsilon}-\frac{1}{2}\xi_j\xi_k\bigg)\bigg|_{+}\,\mathrm{d}s(\xi).
\end{aligned}
$$

From a combination of the above identities we obtain:

$$\begin{aligned}
&\frac{1}{\mu_*}\int_B \operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot\operatorname{grad}\bigg(\hat{v}_{2jk}^{\mu,\varepsilon}-\frac{1}{2}\xi_j\xi_k\bigg)\,\mathrm{d}\xi\\
&\quad+\frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\cdot\operatorname{grad}\bigg(\hat{v}_{2jk}^{\mu,\varepsilon}-\frac{1}{2}\xi_j\xi_k\bigg)\,\mathrm{d}\xi\\
&\;=-\frac{1}{\mu_*}\bigg(\frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}-1\bigg)\delta_{jk}\int_B \big(\hat{v}_{1i}^{\mu}-\xi_i\big)\,\mathrm{d}\xi\\
&\quad+\frac{1}{2}\bigg(\frac{1}{\mu_0}-\frac{1}{\mu_*}\bigg)\int_{\partial B}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\frac{\partial}{\partial\nu}(\xi_j\xi_k)\,\mathrm{d}s(\xi).
\end{aligned}\tag{3.2}$$

The last integral in the right-hand side of (3.2) may be integrated by parts to become:

$$\begin{aligned}
\frac{1}{2}\bigg(\frac{1}{\mu_0}-\frac{1}{\mu_*}\bigg)\bigg[&\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_-(\xi)\xi_j\xi_k\,\mathrm{d}s(\xi)+2\delta_{jk}\int_B\big(\hat{v}_{1i}^{\mu}(\xi)-\xi_i\big)\,\mathrm{d}\xi\\
&-\int_B(\delta_{ij}\xi_k+\delta_{ik}\xi_j)\,\mathrm{d}\xi\bigg].
\end{aligned}$$

Substituting back into (3.2) we finally obtain (c).

To establish (d) we use (2.35) and (2.42) once more to compute:

$$\int_B \operatorname{grad}\big(\hat{v}_{1k}^{\mu}-\xi_k\big)\cdot\operatorname{grad}\bigg(\hat{v}_{2ij}^{\mu,\varepsilon}-\frac{1}{2}\xi_i\xi_j\bigg)\,\mathrm{d}\xi=\int_{\partial B}\bigg(\hat{v}_{2ij}^{\mu,\varepsilon}-\frac{1}{2}\xi_i\xi_j\bigg)\frac{\partial}{\partial\nu}\big(\hat{v}_{1k}^{\mu}-\xi_k\big)\big|_-\,\mathrm{d}s(\xi),$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1k}^{\mu}-\xi_k\big)\cdot\operatorname{grad}\bigg(\hat{v}_{2ij}^{\mu,\varepsilon}-\frac{1}{2}\xi_i\xi_j\bigg)\,\mathrm{d}\xi\\
&\quad=-\int_{\partial B}\bigg(\hat{v}_{2ij}^{\mu,\varepsilon}-\frac{1}{2}\xi_i\xi_j\bigg)\frac{\partial}{\partial\nu}\big(\hat{v}_{1k}^{\mu}-\xi_k\big)\big|_+\,\mathrm{d}s(\xi).
\end{aligned}$$

It follows directly from the jump condition for $\partial\hat{v}_{1k}^{\mu}/\partial\nu$ on $\partial B$ that

$$\begin{aligned}
&\frac{1}{\mu_*}\int_B \operatorname{grad}\big(\hat{v}_{1k}^{\mu}-\xi_k\big)\cdot\operatorname{grad}\bigg(\hat{v}_{2ij}^{\mu,\varepsilon}-\frac{1}{2}\xi_i\xi_j\bigg)\,\mathrm{d}\xi\\
&\quad+\frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1k}^{\mu}-\xi_k\big)\cdot\operatorname{grad}\bigg(\hat{v}_{2ij}^{\mu,\varepsilon}-\frac{1}{2}\xi_i\xi_j\bigg)\,\mathrm{d}\xi
\end{aligned}$$

$$= \left(\frac{1}{\mu_0} - \frac{1}{\mu_*}\right) \int_{\partial B} \left(\hat{v}_{2ij}^{\mu,\varepsilon} - \frac{1}{2}\xi_i \xi_j\right) \nu_k \,\mathrm{d}s(\xi). \tag{3.3}$$

Application of the divergence theorem to $\int_{\partial B} \hat{v}_{2ij}^{\mu,\varepsilon} \nu_k$ yields:

$$\int_{\partial B} \hat{v}_{2ij}^{\mu,\varepsilon}(\xi)\nu_k \,\mathrm{d}s(\xi) = \int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_- (\xi)\xi_k \,\mathrm{d}s(\xi) - \frac{\varepsilon_* \mu_*}{\varepsilon_0 \mu_0} \delta_{ij} \int_B \xi_k \,\mathrm{d}\xi.$$

Substituting back into (3.3) we finally have:

$$\begin{aligned}
&\frac{1}{\mu_*} \int_B \operatorname{grad}\big(\hat{v}_{1k}^{\mu} - \xi_k\big) \cdot \operatorname{grad}\left(\hat{v}_{2ij}^{\mu,\varepsilon} - \frac{1}{2}\xi_i \xi_j\right) \mathrm{d}\xi \\
&\quad + \frac{1}{\mu_0} \int_{\mathbb{R}^d \setminus \overline{B}} \operatorname{grad}\big(\hat{v}_{1k}^{\mu} - \xi_k\big) \cdot \operatorname{grad}\left(\hat{v}_{2ij}^{\mu,\varepsilon} - \frac{1}{2}\xi_i \xi_j\right) \mathrm{d}\xi \\
&\qquad = \left(\frac{1}{\mu_0} - \frac{1}{\mu_*}\right)\left[\int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_- (\xi)\xi_k \,\mathrm{d}s(\xi) - \frac{\varepsilon_* \mu_*}{\varepsilon_0 \mu_0} \delta_{ij} \int_B \xi_k \,\mathrm{d}\xi \right. \\
&\qquad\qquad \left. - \frac{1}{2} \int_{\partial B} \xi_i \xi_j \nu_k \,\mathrm{d}s(\xi)\right].
\end{aligned} \tag{3.4}$$

Since

$$\int_{\partial B} \xi_i \xi_j \nu_k \,\mathrm{d}s(\xi) = \int_B (\xi_i \delta_{jk} + \xi_j \delta_{ik}) \,\mathrm{d}\xi,$$

(3.4) gives (d) exactly as desired. □

### *3.2. Properties of $M^{1,1}$*

We now show that the following properties of the generalized polarization tensor $M^{1,1}$ hold.

**Theorem 3.1.** *The generalized polarization tensor $M^{1,1}$ is symmetric positive definite. Moreover, the diagonal terms*

$$M_{ii}^{1,1} = \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_- (\xi)\xi_i \,\mathrm{d}s(\xi)$$

*satisfy the following optimal bounds*:

- *For* $\mu_*/\mu_0 > 1$

$$|B| \leqslant \int\limits_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_i \, \mathrm{d}s(\xi) \leqslant \frac{\mu_*}{\mu_0}|B|.$$

- *For* $\mu_*/\mu_0 < 1$

$$\frac{\mu_*}{\mu_0}|B| \leqslant \int\limits_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_i \, \mathrm{d}s(\xi) \leqslant |B|.$$

- *For* $\mu_*/\mu_0 = 1$

$$\int\limits_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_i \, \mathrm{d}s(\xi) = |B|.$$

**Proof.** The fact that $M^{1,1}$ is symmetric follows from (a). From (b) we obtain:

$$\left(\frac{1}{\mu_*} + \frac{1}{\mu_0}\right) \sum_{i,j=1}^{d} \left( \int\limits_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-} \xi_j \right) \zeta_i \zeta_j = \frac{1}{\mu_0}|B| \sum_{i=1}^{d} \zeta_i^2 + \frac{1}{\mu_*} \int\limits_{B} |\operatorname{grad} \Phi|^2(\xi) \, \mathrm{d}\xi + \frac{1}{\mu_0} \int\limits_{\mathbb{R}^d \setminus \overline{B}} |\operatorname{grad} \Psi|^2(\xi) \, \mathrm{d}\xi,$$

where $\Phi(\xi) = \sum_{i=1}^{d} \zeta_i \hat{v}_{1i}^{\mu}(\xi)$ and $\Psi(\xi) = \sum_{i=1}^{d} \zeta_i (\hat{v}_{1i}^{\mu}(\xi) - \xi_i)$. Consequently, $M^{1,1}$ is positive definite.

We now establish optimal bounds for the diagonal terms of $M^{1,1}$. Note that $\hat{v}_{1i}^{\mu}(\xi) = \xi_i$ in $\mathbb{R}^d$ if $\mu_* = \mu_0$. Suppose that $\mu_0 \neq \mu_*$. Using (b) once more we get from

$$\begin{aligned} &\int\limits_{B} \left|\tau \operatorname{grad} \hat{v}_{1i}^{\mu}(\xi) + \mathbf{u}_i\right|^2 \mathrm{d}\xi \\ &\quad = \tau^2 \int\limits_{B} \left|\operatorname{grad} \hat{v}_{1i}^{\mu}(\xi)\right|^2 \mathrm{d}\xi + 2\tau \int\limits_{B} \operatorname{grad} \hat{v}_{1i}^{\mu}(\xi) \cdot \mathbf{u}_i + |B| \\ &\quad = \tau^2 \int\limits_{B} \left|\operatorname{grad} \hat{v}_{1i}^{\mu}(\xi)\right|^2 \mathrm{d}\xi + 2\tau \int\limits_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_i \, \mathrm{d}s(\xi) + |B|, \end{aligned}$$

$\forall \tau \in \mathbb{R}$, that we have:

$$\left(\frac{1}{\mu_0}+\frac{1}{\mu_*}+\frac{2}{\tau\mu_*}\right)\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_{-}(\xi)\xi_i\,\mathrm{d}s(\xi)$$
$$=\left(\frac{1}{\mu_0}-\frac{1}{\tau^2\mu_*}\right)|B|+\frac{1}{\tau^2\mu_*}\int_B\left|\tau\,\mathrm{grad}\,\hat{v}_{1i}^{\mu}(\xi)+\mathbf{u}_i\right|^2\mathrm{d}\xi$$
$$+\frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}}\left|\mathrm{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\right|^2\mathrm{d}\xi,\quad\forall\tau\neq 0.$$

Taking $\tau=-1$ in the above identity we arrive at

$$\left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_{-}(\xi)\xi_i\,\mathrm{d}s(\xi)=\left(\frac{1}{\mu_0}-\frac{1}{\mu_*}\right)|B|+\frac{1}{\mu_*}\int_B\left|-\mathrm{grad}\,\hat{v}_{1i}^{\mu}(\xi)+\mathbf{u}_i\right|^2\mathrm{d}\xi$$
$$+\frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}}\left|\mathrm{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\right|^2\mathrm{d}\xi,$$

and therefore

$$|B|\leqslant\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_{-}(\xi)\xi_i\,\mathrm{d}s(\xi)\quad\text{if }\frac{\mu_*}{\mu_0}>1$$

and

$$\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_{-}(\xi)\xi_i\,\mathrm{d}s(\xi)\leqslant|B|\quad\text{if }\frac{\mu_*}{\mu_0}<1.$$

Taking $\tau=-\mu_0/\mu_*$ yields:

$$\left(\frac{1}{\mu_*}-\frac{1}{\mu_0}\right)\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_{-}(\xi)\xi_i\,\mathrm{d}s(\xi)$$
$$=\frac{1}{\mu_0^2}(\mu_0-\mu_*)|B|+\frac{\mu_*}{\mu_0^2}\int_B\left|-\frac{\mu_0}{\mu_*}\,\mathrm{grad}\,\hat{v}_{1i}^{\mu}(\xi)+\mathbf{u}_i\right|^2\mathrm{d}\xi$$
$$+\frac{1}{\mu_0}\int_{\mathbb{R}^d\setminus\overline{B}}\left|\mathrm{grad}\big(\hat{v}_{1i}^{\mu}-\xi_i\big)\right|^2\mathrm{d}\xi.$$

Rewriting the above identity as

$$\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_i \,\mathrm{d}s(\xi) = \frac{\mu_*}{\mu_0}|B| + \frac{\mu_*^2}{(\mu_0-\mu_*)\mu_0}\int_B \left|-\frac{\mu_0}{\mu_*}\operatorname{grad}\hat{v}_{1i}^{\mu}(\xi) + \mathbf{u}_i\right|^2 \mathrm{d}\xi$$
$$+ \frac{\mu_*}{\mu_0-\mu_*}\int_{\mathbb{R}^d\setminus\overline{B}} \left|\operatorname{grad}\left(\hat{v}_{1i}^{\mu} - \xi_i\right)\right|^2 \mathrm{d}\xi,$$

we obtain:

$$\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_i \,\mathrm{d}s(\xi) \leqslant \frac{\mu_*}{\mu_0}|B| \quad \text{if } \frac{\mu_*}{\mu_0} > 1$$

and

$$\frac{\mu_*}{\mu_0}|B| \leqslant \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_i \,\mathrm{d}s(\xi) \quad \text{if } \frac{\mu_*}{\mu_0} < 1.$$

The proof of the theorem is now complete. □

### *3.3. Limits of the generalized polarization tensors corresponding to extreme conductivity cases*

We now study the limits of the generalized polarization tensors $M^{1,1}$, $M^{1,2}$, and $M^{2,1}$ as $\mu_*/\mu_0$ tends towards zero or infinity. To do so, we introduce the functions $\varphi_i, \psi_i, \varphi_{ij}$, and $\psi_{ij}$, $i, j = 1, \ldots, d$, defined in $\mathbb{R}^d \setminus \overline{B}$ and satisfying:

$$\begin{cases} \Delta\varphi_i = 0 & \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \dfrac{\partial \varphi_i}{\partial \nu} = -\nu_i & \text{on } \partial B, \\ \lim\limits_{|\xi|\to+\infty} \varphi_i(\xi) = 0, & \end{cases} \tag{3.5}$$

$$\begin{cases} \Delta\psi_i = 0 & \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \psi_i = -\xi_i + c_i & \text{on } \partial B, \\ \lim\limits_{|\xi|\to+\infty} \psi_i(\xi) = 0, & \end{cases} \tag{3.6}$$

$$\begin{cases} \Delta\varphi_{ij} = 0 & \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \dfrac{\partial \varphi_{ij}}{\partial \nu} = -\dfrac{1}{2}(\nu_i\xi_j + \nu_j\xi_i) & \text{on } \partial B, \\ \lim\limits_{|\xi|\to+\infty} \varphi_{ij}(\xi) + \dfrac{1}{2\pi}|B|\log|\xi|\delta_{ij} = 0, & d = 2, \\ \lim\limits_{|\xi|\to+\infty} \varphi_{ij}(\xi) = 0, & d = 3, \end{cases} \tag{3.7}$$

$$\begin{cases} \Delta\psi_{ij} = 0 & \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \psi_{ij} = -\dfrac{1}{2}\xi_i\xi_j + c_{ij} & \text{on } \partial B, \\ \lim\limits_{|\xi|\to+\infty} \psi_{ij}(\xi) = 0. \end{cases} \tag{3.8}$$

In the boundary conditions (3.6) and (3.8) the constants $c_i$ and $c_{ij}$ are to be chosen such that $\psi_i \to 0$ and $\psi_{ij} \to 0$ as $|\xi| \to +\infty$. The tensor

$$\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\varphi_i(\xi)\cdot\operatorname{grad}\varphi_j(\xi)\,\mathrm{d}\xi$$

is the so-called *virtual mass* and

$$\int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\psi_i(\xi)\cdot\operatorname{grad}\psi_j(\xi)\,\mathrm{d}\xi$$

is the so-called *polarization tensor*. The reader is referred to Schiffer and Szegö [43] and Pólya and Szegö [42] for fundamental studies of these quantities. The following convergences hold for extreme cases. The first two results are first proven in [15] in a slightly different manner while the others have not been established before this work. We emphasize the fact that in order to give physical interpretation to these asymptotic results we should interchange the roles of $\varepsilon$ and $\mu$.

**Theorem 3.2.** *The following convergences hold*:

$$\text{(i)} \int_{\partial B} \frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_{+}(\xi)\xi_j\,\mathrm{d}s(\xi) \to |B|\delta_{ij} + \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\psi_i(\xi)\cdot\operatorname{grad}\psi_j(\xi)\,\mathrm{d}\xi \quad \textit{as } \frac{\mu_*}{\mu_0} \to 0.$$

$$\text{(ii)} \int_{\partial B} \frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_{-}(\xi)\xi_j\,\mathrm{d}s(\xi) \to |B|\delta_{ij} + \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\varphi_i(\xi)\cdot\operatorname{grad}\varphi_j(\xi)\,\mathrm{d}\xi \quad \textit{as } \frac{\mu_*}{\mu_0} \to +\infty.$$

$$\begin{aligned} \text{(iii)} \int_{\partial B} \frac{\partial\hat{v}_{2ij}^{\mu,\varepsilon}}{\partial\nu}\bigg|_{+}(\xi)\xi_k\,\mathrm{d}s(\xi) &\to \delta_{ij}\int_B \xi_k\,\mathrm{d}\xi + \frac{1}{2}\int_B (\xi_i\delta_{jk} + \xi_j\delta_{ik})\,\mathrm{d}\xi \\ &\quad + \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\psi_k(\xi)\cdot\operatorname{grad}\psi_{ij}(\xi)\,\mathrm{d}\xi \quad \textit{as } \frac{\mu_*}{\mu_0} \to 0 \textit{ and } \frac{\varepsilon_*}{\varepsilon_0} \to 1. \end{aligned}$$

(iv) $$\int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \nu}\bigg|_{-}(\xi)\xi_k \,\mathrm{d}s(\xi)$$

$$\to \delta_{ij}\int_B \xi_k \,\mathrm{d}\xi + \frac{1}{2}\int_B (\xi_i\delta_{jk} + \xi_j\delta_{ik})\,\mathrm{d}\xi + \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\varphi_i(\xi)\cdot\operatorname{grad}\varphi_{jk}(\xi)\,\mathrm{d}\xi$$

*as* $\frac{\mu_*}{\mu_0} \to +\infty$ *and* $\frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0} \to 1$.

(v) $$\frac{1}{2}\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{+}(\xi)\xi_j\xi_k \,\mathrm{d}s(\xi) \to \frac{1}{2}\int_B (\xi_k\delta_{ij} + \xi_j\delta_{ik})\,\mathrm{d}\xi + \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\psi_i(\xi)\cdot\operatorname{grad}\psi_{jk}(\xi)\,\mathrm{d}\xi \quad \textit{as } \frac{\mu_*}{\mu_0}\to 0.$$

(vi) $$\frac{1}{2}\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi)\xi_j\xi_k \,\mathrm{d}s(\xi)$$

$$\to \frac{1}{2}\int_B (\xi_k\delta_{ij} + \xi_j\delta_{ik})\,\mathrm{d}\xi + \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\varphi_i(\xi)\cdot\operatorname{grad}\varphi_{jk}(\xi)\,\mathrm{d}\xi$$

*as* $\frac{\mu_*}{\mu_0} \to +\infty$ *and* $\frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0} \to 1$.

**Proof.** We first recall that

$$\hat{v}_{1i}^{\mu}(\xi) = \xi_i + \left(1 - \frac{\mu_0}{\mu_*}\right)\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi')G^0(\xi,\xi')\,\mathrm{d}(\xi'), \quad \xi\in\mathbb{R}^d\setminus\overline{B}.$$

Let $X$ be the space of functions $X = \{p\in L^2(\partial B)\colon \int_{\partial B} p = 0\}$. From classical potential theory it follows that $\partial\hat{v}_{1i}^{\mu}/\partial\nu|_+$ is a solution in $X$ to the integral equation:

$$\frac{1}{2}\left(1 + \frac{\mu_*}{\mu_0}\right)\frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{+}(\xi) + \left(1 - \frac{\mu_*}{\mu_0}\right)\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{+}(\xi')\frac{\partial G^0}{\partial\nu(\xi)}(\xi,\xi')\,\mathrm{d}s(\xi') = \nu_i, \quad \forall\xi\in\partial B.$$

Similarly using the jump condition for $\partial\hat{v}_{1i}^{\mu}/\partial\nu$ on $\partial B$ we obtain that $\partial\hat{v}_{1i}^{\mu}/\partial\nu|_-$ solves:

$$\frac{1}{2}\frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi) + \frac{1-\mu_*/\mu_0}{1+\mu_*/\mu_0}\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_{-}(\xi')\frac{\partial G^0}{\partial\nu(\xi)}(\xi,\xi')\,\mathrm{d}s(\xi') = \frac{\mu_*}{\mu_*+\mu_0}\nu_i, \quad \forall\xi\in\partial B.$$

Writing

$$\psi(\xi) = -\int_{\partial B} p_i(\xi') G^0(\xi, \xi')\, \mathrm{d}s(\xi')$$

as a single layer potential with density $p_i \in X$, a necessary and sufficient condition that $\psi$ solves (3.6) is:

$$\frac{1}{2} p_i(\xi) + \int_{\partial B} p_i(\xi') \frac{\partial G^0}{\partial \nu(\xi)}(\xi, \xi')\, \mathrm{d}s(\xi') = \nu_i, \quad \forall \xi \in \partial B. \tag{3.9}$$

Similarly writing

$$\varphi_i(\xi) = \int_{\partial B} q_i(\xi') G^0(\xi, \xi')\, \mathrm{d}s(\xi')$$

as a single layer potential with density $q_i \in X$ we obtain that $\varphi_i$ is a solution to (3.5) iff

$$\frac{1}{2} q_i(\xi) - \int_{\partial B} q_i(\xi') \frac{\partial G^0}{\partial \nu(\xi)}(\xi, \xi')\, \mathrm{d}s(\xi') = \nu_i, \quad \forall \xi \in \partial B. \tag{3.10}$$

From the uniqueness associated with (3.6) it follows that (3.9) has a unique solution. By the compactness of the operator,

$$p \in X \mapsto \int_{\partial B} p_i(\xi') \frac{\partial G^0}{\partial \nu(\xi)}(\xi, \xi')\, \mathrm{d}s(\xi') \in X,$$

it follows that $\partial \hat{v}_{1i}^{\mu} / \partial \nu|_+$ converges uniformly to $p_i$ as $\mu_* / \mu_0 \to 0$ and consequently,

$$\hat{v}_{1i}^{\mu}(\xi) - \xi_i \to -\int_{\partial B} p_i(\xi') G^0(\xi, \xi')\, \mathrm{d}s(\xi') = \psi_i(\xi)$$

in any compact subset of $\mathbb{R}^d \setminus \overline{B}$. Actually, we obtain even better uniform convergence by application of the maximum principle. Since

$$\begin{aligned}
&\Delta\big(\hat{v}_{1i}^{\mu} - \xi_i - \psi_i\big) = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B},\\
&\big(\hat{v}_{1i}^{\mu} - \xi_i - \psi_i\big) \to 0 \quad \text{uniformly on } \partial B,\\
&\lim_{|\xi| \to +\infty} \big(\hat{v}_{1i}^{\mu} - \xi_i - \psi_i\big)(\xi) = 0,
\end{aligned}$$

by the maximum principle it follows that the harmonic function $\hat{v}_{1i}^{\mu} - \xi_i - \psi_i$ converges uniformly to zero in $\mathbb{R}^d \setminus \overline{B}$. Rewriting

$$\frac{\mu_0}{\mu_*}\int_B \operatorname{grad}\hat{v}_{1i}^{\mu}(\xi)\cdot\operatorname{grad}\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}\xi = \frac{\mu_0}{\mu_*}\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_-(\xi)\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}s(\xi)$$
$$= \int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_+(\xi)\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}s(\xi),$$

and noting that

$$\int_{\partial B}\frac{\partial\hat{v}_{1i}^{\mu}}{\partial\nu}\bigg|_+(\xi)\hat{v}_{1j}^{\mu}(\xi)\,\mathrm{d}s(\xi)\to -c_j\int_{\partial B}p_i(\xi)\,\mathrm{d}s(\xi)=0,$$

the statement (i) now follows from identity (b).

We now prove (ii). By an argument similar to that used for showing (i) we can prove that $\partial\hat{v}_{1i}^{\mu}/\partial\nu|_-(\xi)$ converges uniformly to $q_i(\xi)$ as $\mu_*/\mu_0\to+\infty$ and therefore, by the maximum principle $\hat{v}_{1i}^{\mu}(\xi)-\xi_i$ converges uniformly to $\varphi_i(\xi)$ in all of $\mathbb{R}^d\setminus\overline{B}$. The statement (ii) immediately follows from identity (c).

To prove (iii) and (iv) we recall that

$$\hat{v}_{2ij}^{\mu,\varepsilon}(\xi) = \left(\frac{\mu_*}{\mu_0}-1\right)\int_{\partial B}\frac{\partial\hat{v}_{2ij}^{\mu,\varepsilon}}{\partial\nu}\bigg|_+(\xi')G^0(\xi,\xi')\,\mathrm{d}s(\xi')$$
$$+\frac{1}{2}\xi_i\xi_j+\delta_{ij}\left(1-\frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}\right)\int_B G^0(\xi,\xi')\,\mathrm{d}\xi'.$$

This integral representation formula together with

$$\int_{\partial B}\frac{\partial\hat{v}_{2ij}^{\mu,\varepsilon}}{\partial\nu}\bigg|_+(\xi)\,\mathrm{d}s(\xi) = \frac{\mu_0}{\mu_*}\int_{\partial B}\frac{\partial\hat{v}_{2ij}^{\mu,\varepsilon}}{\partial\nu}\bigg|_-(\xi)\,\mathrm{d}s(\xi) = \frac{\varepsilon_*}{\varepsilon_0}|B|\delta_{ij},$$

yield the fact that $\partial\hat{v}_{2ij}^{\mu,\varepsilon}/\partial\nu|_+$ is the unique solution in

$$\left\{p\in L^2(\partial B)\colon \int_{\partial B}p=\frac{\varepsilon_*}{\varepsilon_0}|B|\delta_{ij}\right\}$$

to the integral equation,

$$\frac{1}{2}\left(1+\frac{\mu_*}{\mu_0}\right)\frac{\partial\hat{v}_{2ij}^{\mu,\varepsilon}}{\partial\nu}\bigg|_+(\xi)+\left(1-\frac{\mu_*}{\mu_0}\right)\int_{\partial B}\frac{\partial\hat{v}_{2ij}^{\mu,\varepsilon}}{\partial\nu}\bigg|_+(\xi')\frac{\partial G^0}{\partial\nu(\xi)}(\xi,\xi')\,\mathrm{d}s(\xi')$$
$$=\frac{1}{2}(\nu_i\xi_j+\nu_j\xi_i)+\delta_{ij}\left(1-\frac{\varepsilon_*}{\varepsilon_0}\right)\int_B\frac{\partial G^0}{\partial\nu(\xi)}(\xi,\xi')\,\mathrm{d}\xi',\quad \xi\in\partial B.$$

It is also easy to verify that $\partial\hat{v}^{\mu,\varepsilon}_{2ij}/\partial\nu|_-$ is the unique solution in

$$\left\{q \in L^2(\partial B)\colon \int_{\partial B} q = \frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}|B|\delta_{ij}\right\}$$

to the integral equation,

$$\frac{1}{2}\left(1+\frac{\mu_0}{\mu_*}\right)\frac{\partial\hat{v}^{\mu,\varepsilon}_{2ij}}{\partial\nu}\bigg|_-(\xi) - \left(1-\frac{\mu_0}{\mu_*}\right)\int_{\partial B}\frac{\partial\hat{v}^{\mu,\varepsilon}_{2ij}}{\partial\nu}\bigg|_-(\xi')\frac{\partial G^0}{\partial\nu(\xi)}(\xi,\xi')\,\mathrm{d}s(\xi')$$
$$= \frac{1}{2}(\nu_i\xi_j+\nu_j\xi_i) + \delta_{ij}\left(1-\frac{\varepsilon_*\mu_*}{\varepsilon_0\mu_0}\right)\int_B \frac{\partial G^0}{\partial\nu(\xi)}(\xi,\xi')\,\mathrm{d}\xi', \quad \xi\in\partial B.$$

To show (iii) let $p_{ij} \in \{p \in L^2(\partial B)\colon \int_{\partial B} p = |B|\delta_{ij}\}$ denote the unique solution to the integral equation,

$$\frac{1}{2}p_{ij}(\xi) + \int_{\partial B} p_{ij}(\xi')\frac{\partial G^0}{\partial\nu(\xi)}(\xi,\xi')\,\mathrm{d}s(\xi')$$
$$= \frac{1}{2}(\nu_i\xi_j+\nu_j\xi_i) + \delta_{ij}\int_B \frac{\partial G^0}{\partial\nu(\xi)}(\xi,\xi')\,\mathrm{d}\xi', \quad \xi\in\partial B.$$

We may show by the same arguments as those for proving (i) that $\partial\hat{v}^{\mu,\varepsilon}_{2ij}/\partial\nu|_+(\xi)$ converges uniformly to $p_{ij}(\xi)$ as $\mu_*/\mu_0 \to 0$ and $\varepsilon_*/\varepsilon_0 \to 1$. From

$$\int_{\partial B}\frac{\partial}{\partial\nu}\left(\hat{v}^{\mu,\varepsilon}_{2ij} - \frac{1}{2}\xi_i\xi_j\right)\bigg|_+\big(\xi_k - \hat{v}^{\mu}_{1k}\big)\,\mathrm{d}(\xi)$$
$$= \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\left(\hat{v}^{\mu,\varepsilon}_{2ij} - \frac{1}{2}\xi_i\xi_j\right)\cdot\operatorname{grad}\big(\hat{v}^{\mu}_{1k} - \xi_k\big)\,\mathrm{d}\xi,$$

together with

$$\int_{\partial B}\frac{\partial\hat{v}^{\mu,\varepsilon}_{2ij}}{\partial\nu}\bigg|_+(\xi)\hat{v}^{\mu}_{1k}(\xi)\,\mathrm{d}s(\xi) \to c_k|B|\delta_{ij},$$

and

$$\frac{1}{2}\int_{\partial B}\frac{\partial}{\partial\nu}(\xi_i\xi_j)\hat{v}^{\mu}_{1k}(\xi)\,\mathrm{d}(\xi) \to c_k|B|\delta_{ij},$$

it immediately follows that statement (iii) holds.

In a fashion completely similar to that in (ii) we represent $\varphi_{ij}$ in the form of a single layer potential:

$$\varphi_{ij}(\xi) = \int_{\partial B} q_{ij}(\xi') G^0(\xi, \xi') \, \mathrm{d}s(\xi'),$$

where the density $q_{ij} \in \{q \in L^2(\partial B): \int_{\partial B} q = |B|\delta_{ij}\}$, and prove that $\partial \hat{v}_{2ij}^{\mu,\varepsilon}/\partial\nu|_-$ converges uniformly to $q_{ij}$ and so, by the maximum principle,

$$\hat{v}_{2ij}^{\mu,\varepsilon}(\xi) \to \frac{1}{2}\xi_i\xi_j + \varphi_{ij}(\xi) \quad \text{in } \mathbb{R}^d \setminus \overline{B},$$

as $\mu_*/\mu_0 \to +\infty$ and $\varepsilon_*\mu_*/\varepsilon_0\mu_0 \to 1$, which yields the statement (iv).

Note that from

$$\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_+ (\xi) \, \mathrm{d}s(\xi) = 0$$

it follows that

$$\frac{1}{2}\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_+ (\xi)\xi_j\xi_k \, \mathrm{d}s(\xi) = \int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_+ (\xi)\left(\frac{1}{2}\xi_j\xi_k - c_{ij}\right) \mathrm{d}s(\xi),$$

where the constant $c_{ij}$ is such that $\psi_{ij} \to 0$ as $|\xi| \to +\infty$. Integrating by parts we obtain:

$$\frac{1}{2}\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_+ (\xi)\xi_j\xi_k \, \mathrm{d}s(\xi) = \frac{1}{2}\int_B (\xi_k\delta_{ij} + \xi_j\delta_{ik}) \, \mathrm{d}\xi + \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\big(\hat{v}_{1i}^{\mu} - \xi_i\big) \cdot \operatorname{grad}\psi_{jk}(\xi) \, \mathrm{d}\xi$$

and therefore

$$\frac{1}{2}\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_+ (\xi)\xi_j\xi_k \, \mathrm{d}s(\xi) \to \frac{1}{2}\int_B (\xi_k\delta_{ij} + \xi_j\delta_{ik}) \, \mathrm{d}\xi + \int_{\mathbb{R}^d\setminus\overline{B}} \operatorname{grad}\psi_i(\xi) \cdot \operatorname{grad}\psi_{jk}(\xi) \, \mathrm{d}\xi,$$

as $\mu_*/\mu_0 \to 0$.

We finally prove (vi). From $\partial \hat{v}_{1i}^{\mu}/\partial\nu|_-(\xi)$ converges uniformly to $q_i(\xi)$ and

$$\int_{\partial B} \frac{\partial \hat{v}_{1i}^{\mu}}{\partial \nu}\bigg|_- (\xi) \, \mathrm{d}s(\xi) = 0$$

it is easy to prove that

$$\hat{v}_{1i}^{\mu}(\xi) \to \xi_i \quad \text{in } B \text{ as } \frac{\mu_*}{\mu_0} \to +\infty.$$

From the fact that $\hat{v}_{1i}^{\mu}(\xi) - \xi_i$ converges uniformly to $\varphi_i(\xi)$ in all of $\mathbb{R}^d \setminus \overline{B}$ as $\mu_*/\mu_0 \to +\infty$ and

$$\hat{v}_{2ij}^{\mu,\varepsilon}(\xi) \to \frac{1}{2}\xi_i \xi_j + \varphi_{ij}(\xi) \quad \text{in } \mathbb{R}^d \setminus \overline{B},$$

as $\mu_*/\mu_0 \to +\infty$ and $\varepsilon_* \mu_*/(\varepsilon_0 \mu_0) \to 1$, we conclude by using identity (c) that (vi) holds. The proof of the theorem is now complete. □

## 4. Resonant frequencies

In this section we provide a rigorous derivation of an asymptotic formula for perturbations in the resonant frequencies caused by the presence of a finite number of dielectric inhomogeneities of small diameter, see [19,30,48] for related work.

The eigenvalue problem in the presence of imperfections consists of finding $\omega_\alpha$ such that there exists a nontrivial electric field $E_\alpha$ that is solution to,

$$\begin{aligned}
&\operatorname{div}\left(\frac{1}{\mu_\alpha} \operatorname{grad} E_\alpha\right) + \omega_\alpha^2 \varepsilon_\alpha E_\alpha = 0 \quad \text{in } \Omega,\\
&E_\alpha = 0 \quad \text{or} \quad \frac{\partial E_\alpha}{\partial \nu} = 0 \quad \text{on } \partial\Omega,\\
&\int_\Omega \varepsilon_\alpha E_\alpha^2 = 1.
\end{aligned} \tag{4.1}$$

It is well known that all eigenvalues of (4.1) are real, of finite multiplicity, have no finite accumulation points, and there are corresponding eigenfunctions which make up an orthonormal basis of $L^2(\Omega)$.

Let $\omega_0$ be an eigenvalue of multiplicity $n$ for the Helmholtz equation in the absence of any inhomogeneities. Then there exist $n$ nonzero solutions $\{E_0^i\}_{i=1}^n$ to,

$$\begin{aligned}
&\Delta E_0^i + \omega_0^2 \mu_0 \varepsilon_0 E_0^i = 0 \quad \text{in } \Omega,\\
&E_0^i = 0 \text{ or } \frac{\partial E_0^i}{\partial \nu} = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{4.2}$$

such that

$$\int_\Omega \varepsilon_0 E_0^i \cdot E_0^l = \delta_{il}, \quad \forall i, l = 1, \ldots, n, \tag{4.3}$$

where $\delta_{il}$ is the Kronecker symbol.

Our main results in this section are summarized in the following theorem:

**Theorem 4.1.** *Suppose $\omega_0$ is eigenfrequency of multiplicity $n$ for the Helmholtz equation in absence of inhomogeneities, and let $\{E_0^i\}_{i=1}^n$ denote the corresponding eigenfunctions defined in* (4.2)–(4.3). *For $\alpha$ small enough there exist $n$ eigenfrequencies $\{\omega_\alpha^i\}_{i=1}^n$* (*counted according to multiplicity*) *for the Helmholtz equation* (4.1) *in presence of small imperfections that converge to $\omega_0$ as $\alpha$ approaches zero. Then the following asymptotic formula pertaining to the convergence in average of the inverse of the squared eigenvalues of the perturbed problem, that is the convergence of $1/n\sum_{i=1}^n 1/(\omega_\alpha^i)^2$ holds*:

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^n \frac{1}{(\omega_\alpha^i)^2} - \frac{1}{\omega_0^2} = {}& \alpha^d \frac{1}{\omega_0^4\varepsilon_0\mu_0}\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^m\biggl[\left(1-\frac{\mu_0}{\mu_j}\right)\partial_l E_0^i(z_j) M_{lp}^{1,1}\partial_p E_0^i(z_j) \\
&\qquad + \omega_0^2\varepsilon_0\mu_0\left(\frac{\varepsilon_j}{\varepsilon_0}-1\right)|B_j|\bigl(E_0^i(z_j)\bigr)^2\biggr] \\
&+ \alpha^{d+1}\frac{1}{\omega_0^4\mu_0\varepsilon_0}\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^m\biggl[\left(1-\frac{\mu_0}{\mu_j}\right)\bigl(\partial_{lp}^2 E_0^i(z_j) M_{lpq}^{2,1}\partial_q E_0^i(z_j) \\
&\qquad - \partial_l E_0^i(z_j) M_{lpq}^{1,2}\partial_{pq}^2 E_0^i(z_j)\bigr) \\
&+ \omega_0^2\varepsilon_0\mu_0\left(\frac{\varepsilon_j}{\varepsilon_0}-1\right)\partial_l E_0^i(z_j)\int_B \bigl(\hat{v}_{1l}^\mu(\xi)+\xi_l\,\mathrm{d}\xi\bigr)E_0^i(z_j)\biggr] \\
&+ \mathrm{O}\bigl(\alpha^{d+2}\bigr).
\end{aligned}
\tag{4.4}
$$

*The term $\mathrm{O}(\alpha^{d+2})$ is bounded by $C\alpha^{d+2}$, where the constant $C$ depends on the resonant frequency $\omega_0$, the domains $\{B_j\}_{j=1}^m$, $\Omega$, the constants $c_0$, $\{\mu_j,\varepsilon_j\}_{j=0}^m$, but is otherwise independent of the location of the set of points $\{z_j\}_{j=1}^m$.*

### *4.1. Formal derivations*

We restrict the formal derivation to the case of a single inhomogeneity ($m=1$). In the general case we iterate the argument, adding one inhomogeneity at a time. *Assume that $\omega_0$ is simple*. This nondegeneracy condition is essential in the formal derivations. Let for simplicity *assume that* $d=3$. Let $E_0$ denote the corresponding eigenfunction. Let $\lambda_0=\omega_0^2\varepsilon_0\mu_0$, $\lambda_*=\omega_0^2\varepsilon_*\mu_*$, and $\lambda_\alpha=\omega_\alpha^2\varepsilon_0\mu_0$. We seek a solution of (4.1) for $\alpha$ small, for which $\omega_\alpha\to\omega_0$ as $\alpha\to 0$. In this section the eigenvalue $\omega_\alpha$ for the perturbed eigenvalue problem (4.1) is constructed by the method of matched asymptotic expansions for $\alpha$ small. The expansion of $\omega_\alpha$ must begin with $\omega_0$, and the outer expansion of $E_\alpha$ must begin with $E_0$; so we write:

$$\lambda_\alpha = \lambda_0 + \alpha^d\lambda_1 + \alpha^{d+1}\lambda_2 + \cdots, \tag{4.5}$$

$$E_\alpha(y) = E_0(y) + \alpha^d E_1(y) + \alpha^{d+1}E_2(y) + \cdots, \quad \text{for } |y-z| \gg \mathrm{O}(\alpha), \tag{4.6}$$

where $E_1, E_2, \ldots$ are to be found.

Now we substitute (4.5) and (4.6) into the Helmholtz equation (4.1) and equate terms of each power in $\alpha$. The terms of order $\alpha^{d+j-1}$, $j = 1, 2, \ldots,$ yield:

$$(\Delta + \lambda_0)E_j = -\left(\lambda_j E_0 + \sum_{l=1}^{j-d} \lambda_l E_{j+1-l-d}\right), \quad \text{for } |y - z| \gg \mathrm{O}(\alpha),$$
$$E_j = 0 \quad \text{or} \quad \frac{\partial E_j}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Since the behavior of $E_j$ as $y$ tends to $z$ is not yet known, the function $E_j$ is not yet completely defined. However, $E_0$ is the eigenvalue of the unperturbed problem, so it is regular at $z$.

To write the inner expansion of $E_\alpha$, we introduce the stretched variable $\xi = (y - z)/\alpha$ and set $e_\alpha(\xi) = E_\alpha(z + \alpha\xi)$. Then (4.1) becomes:

$$\mathrm{div}_\xi\left(\frac{1}{\mu(\xi)} \mathrm{grad}_\xi\, e_\alpha\right)(\xi) + \alpha^2 \omega_\alpha^2 \varepsilon(\xi) e_\alpha(\xi) = 0,$$

where the stretched coefficients $\varepsilon$ and $\mu$ are defined by (2.30) and (2.31).

Now we write the inner expansion of $E_\alpha$ as

$$E_\alpha(z + \alpha\xi) = e_\alpha(\xi) = e_0(\xi) + \alpha e_1(\xi) + \alpha^2 e_2(\xi) + \cdots, \quad \text{for } |\xi| = \mathrm{O}(1), \tag{4.7}$$

where $e_0, e_1, e_2, \ldots$ are to be found. Next we substitute (4.5) and (4.7) into (4.1) to get:

$$\mathrm{div}\, \frac{1}{\mu}\, \mathrm{grad}\, e_0 = 0, \qquad \mathrm{div}\, \frac{1}{\mu}\, \mathrm{grad}\, e_1 = 0,$$
$$\mathrm{div}\, \frac{1}{\mu}\, \mathrm{grad}\, e_{j+2} = -\frac{\varepsilon}{\mu_0 \varepsilon_0}\left(\lambda_0 e_j + \sum_{l=1}^{j+1-d} \lambda_l e_{j-d-l+1}\right), \quad j = 0, 1, \ldots. \tag{4.8}$$

The matching conditions are:

$$E_0(y) + \alpha^d E_1(y) + \alpha^{d+1} E_2(y) + \cdots \sim e_0(\xi) + \alpha e_1(\xi) + \alpha^2 e_2(\xi) + \cdots. \tag{4.9}$$

From the terms of order $\alpha^0$ in (4.9), we obtain the first matching condition, $e_0(\xi) \to E_0(z)$ as $|\xi| \to +\infty$. Therefore,

$$e_0(\xi) = E_0(z). \tag{4.10}$$

The terms of order $\alpha^j$, $j = 1, 2,$ in (4.9) yield:

$$e_1(\xi) - \partial_i E_0(z)\xi_i \to 0 \quad \text{as } |\xi| \to +\infty,$$

and

$$e_2(\xi) \sim \begin{cases} \dfrac{1}{2}\xi_i\xi_j\partial^2_{ij}E_0(z) + \dfrac{1}{2\pi}\lambda_0|B|\left(1-\dfrac{\varepsilon_*}{\varepsilon_0}\right)\log|\xi|E_0(z) & \text{if } d=2,\\ \dfrac{1}{2}\xi_i\xi_j\partial^2_{ij}E_0(z) & \text{if } d=3, \end{cases}$$

as $|\xi| \to +\infty$. Thus,

$$e_1(\xi) = \partial_i E_0(z)\hat{v}^{\mu}_{1i}(\xi), \tag{4.11}$$

where $\hat{v}^{\mu}_{1i}$ is the unique solution to (2.35) and

$$e_2(\xi) = \partial_{ij} E_0(z)\hat{v}^{\mu,\varepsilon}_{2ij}(\xi), \tag{4.12}$$

where $\hat{v}^{\mu,\varepsilon}_{2ij}$ is the unique solution to (2.42). We now use (4.10)–(4.12) in (4.9) and set $\xi = (y-z)/\alpha$. Then the second term in the far field form of $\partial_i E_0(z)\hat{v}^{\mu}_{1i}(\xi)$ becomes:

$$\left(1-\frac{\mu_0}{\mu_*}\right)\partial_i E_0(z) M^{1,1}_{ij}\partial_j G^0(y,z),$$

where $\partial_j$ denotes the first derivatives in $z$, and the second term in the far field expansion of $\partial_{ij}E_0(z)\hat{v}^{\mu,\varepsilon}_{2ij}(\xi)$ becomes:

$$-\lambda_0\left(1-\frac{\varepsilon_*}{\varepsilon_0}\right)E_0(z)|B|G^0(y,z)$$

which must be matched by the second term $\alpha^d E_1(y)$ in (4.6). Here $G^0$ is defined by (2.19).

Therefore

$$E_1(y) \sim -\lambda_0\left(1-\frac{\varepsilon_*}{\varepsilon_0}\right)|B|E_0(z)G^0(y,z) + \left(1-\frac{\mu_0}{\mu_*}\right)\partial_i E_0(z)M^{1,1}_{ij}\partial_j G^0(y,z) \quad \text{as } y \to z. \tag{4.13}$$

This condition determines the previously unknown behavior of $E_1$ as $y$ tends to $z$.

To find the first correction $\lambda_1$ we multiply $(\Delta+\lambda_0)E_1 = -\lambda_1 E_0$ by $E_0$ and integrate the resulting equation over the region outside a small ball $B_\delta$ of radius $\delta$ centered at $z$. Upon using Green's theorem, we obtain:

$$-\lambda_1 \int_{\Omega\setminus\overline{B_\delta}} E_0^2(y)\,\mathrm{d}y = \int_{\partial B_\delta}\left(E_0\frac{\partial E_1}{\partial\nu} - E_1\frac{\partial E_0}{\partial\nu}\right)\mathrm{d}s(y). \tag{4.14}$$

Here $\partial/\partial\nu$ corresponds to the outward normal derivative to $\Omega\setminus\overline{B_\delta}$.

Now by using the behavior (4.13) of $E_1$ near $z$, we can evaluate the limit of the right-hand side of (4.14) as the radius $\delta$ tends to zero. The integral on the left-hand side tends to $1/\varepsilon_0$ because $E_0$ is normalized, and we obtain from

$$\lim_{\delta\to 0}\int_{\partial B_\delta}\left(E_0\frac{\partial G^0}{\partial\nu}-G^0\frac{\partial E_0}{\partial\nu}\right)\mathrm{d}s(y)=-E_0(z),$$

and

$$\lim_{\delta\to 0}\int_{\partial B_\delta}\left(E_0\frac{\partial}{\partial\nu}\partial_j G^0-\partial_j G^0\frac{\partial E_0}{\partial\nu}\right)\mathrm{d}s(y)=\partial_j E_0(z),$$

that $\lambda_1$ is given by

$$\lambda_1=\frac{1}{\varepsilon_0}\left[\lambda_0\left(1-\frac{\varepsilon_*}{\varepsilon_0}\right)|B|E_0^2(z)-\left(1-\frac{\mu_0}{\mu_*}\right)\partial_i E_0(z)M_{ij}^{1,1}\partial_j E_0(z)\right]. \tag{4.15}$$

Now we extend these calculations to obtain the second correction $\lambda_2$. In (4.6) the third term $\alpha^{d+1}E_2$ must be matched by the sum of the term of order $\mathrm{O}(1/|\xi|^3)$ in the far field of $\partial_i E_0(z)\hat{v}_{1i}^{\mu}(\xi)$, the term of order $\mathrm{O}(1/|\xi|^2)$ in the far field of $\partial_{ij}E_0(z)\hat{v}_{2ij}^{\mu,\varepsilon}(\xi)$, and the term of order $\mathrm{O}(1/|\xi|)$ in the far field of $e_3(\xi)$. By proceeding as before and using (4.14), with $E_1,\lambda_1$, replaced by $E_2,\lambda_2$, we obtain:

$$-\lambda_2=\lim_{\delta\to 0}\int_{\partial B_\delta}\left(E_0\frac{\partial E_2}{\partial\nu}-E_2\frac{\partial E_0}{\partial\nu}\right)\mathrm{d}s(y). \tag{4.16}$$

We now must determine the first term in the far field of $e_3(\xi)$. From (4.8) it follows that

$$\operatorname{div}\frac{1}{\mu(\xi)}\operatorname{grad}e_3(\xi)=-\frac{\lambda_0}{\varepsilon_0\mu_0}\partial_i E_0(z)\varepsilon(\xi)\hat{v}_{1i}^{\mu}(\xi)\quad\text{in }\mathbb{R}^d.$$

From Lemma 2.4 on the behavior of $\hat{v}_{1i}^{\mu}(\xi)$ as $|\xi|\to+\infty$ we know that

$$\hat{v}_{1i}^{\mu}(\xi)=\xi_i+\frac{c_{ij}\xi_j}{|\xi|^3}+\frac{c_i}{|\xi|^3}+\frac{c_{ijk}\xi_j\xi_k}{|\xi|^5}+\mathrm{O}\left(\frac{1}{|\xi|^4}\right), \tag{4.17}$$

as $|\xi|\to+\infty$. Observe that

$$\begin{aligned}
&\Delta\left(\frac{1}{6}\xi_i^2\xi_j\partial_{iij}^3E_0(z)\right)=-\lambda_0\xi_j\partial_j E_0(z),\\
&\Delta\left(\frac{\xi_j}{|\xi|}\right)=-2\frac{\xi_j}{|\xi|^3},\quad \xi\neq 0,\\
&\Delta\left(\frac{\log|\xi|}{|\xi|}\right)=-\frac{1}{|\xi|^3},\quad \xi\neq 0,
\end{aligned}$$

and

$$\Delta\left(\partial_{jk}\log|\xi| - \delta_{jk}\frac{\log|\xi|}{|\xi|}\right) = -3\frac{\xi_j\xi_k}{|\xi|^5}, \quad \xi \neq 0.$$

Then from the integral representation of $e_3(\xi)$ for $\xi \in \mathbb{R}^d \setminus \overline{B}$:

$$\begin{aligned}
e_3(\xi) &= \frac{1}{6}\xi_i\xi_j\xi_k\partial^3_{ijk}E_0(z) - \frac{1}{2}\lambda_0 c_{ij}\frac{\xi_j}{|\xi|}\partial_i E_0(z) \\
&\quad + \frac{\lambda_0}{3}c_{ijk}\left(\partial_{jk}\log|\xi| - \delta_{jk}\frac{\log|\xi|}{|\xi|}\right)\partial_i E_0(z) + \lambda_0 c_i\frac{\log|\xi|}{|\xi|}\partial_i E_0(z) \\
&\quad + (\lambda_0 - \lambda_*)\int_B e_1(\xi')G^0(\xi,\xi')\,\mathrm{d}\xi' - \left(1 - \frac{\mu_0}{\mu_*}\right)\int_{\partial B}\frac{\partial e_3}{\partial\nu}\bigg|_-(\xi')G^0(\xi,\xi')\,\mathrm{d}s(\xi') \\
&\quad - \lambda_0\int_{\mathbb{R}^d\setminus\overline{B}}\left(e_1(\xi') - \left(\xi'_i + c_{ij}\frac{\xi'_j}{|\xi'|^3} + \frac{c_{ij}\xi'_j}{|\xi'|^3} + \frac{c_i}{|\xi'|^3}\right.\right. \\
&\qquad\qquad \left.\left. + \frac{c_{ijk}\xi'_j\xi'_k}{|\xi'|^5}\right)\partial_i E_0(z)\right)G^0(\xi,\xi')\,\mathrm{d}\xi' \\
&\quad + \int_{\partial B}\frac{\partial}{\partial\nu(\xi')}\left(e_3(\xi') - \frac{1}{6}\xi'_i\xi'_j\xi'_k\partial^3_{ijk}E_0(z) + \frac{1}{2}\lambda_0 c_{ij}\frac{\xi'_j}{|\xi'|}\partial_i E_0(z)\right. \\
&\qquad\qquad - \frac{\lambda_0}{3}c_{ijk}\left(\partial_{jk}\log|\xi'| - \delta_{jk}\frac{\log|\xi'|}{|\xi'|}\right)\partial_i E_0(z) \\
&\qquad\qquad \left. - \lambda_0 c_i\frac{\log|\xi'|}{|\xi'|}\partial_i E_0(z)\right)\bigg|_+(\xi')G^0(\xi,\xi'\Big)\,\mathrm{d}\xi',
\end{aligned}$$

together with the identity:

$$\begin{aligned}
&\int_{\partial B}\frac{\partial}{\partial\nu(\xi')}\left(e_3(\xi') - \frac{1}{6}\xi'_i\xi'_j\xi'_k\partial^3_{ijk}E_0(z) + \frac{1}{2}\lambda_0 c_{ij}\frac{\xi'_j}{|\xi'|}\partial_i E_0(z)\right. \\
&\qquad - \frac{\lambda_0}{3}c_{ijk}\left(\partial_{jk}\log|\xi'| - \delta_{jk}\frac{\log|\xi'|}{|\xi'|}\right)\partial_i E_0(z) \\
&\qquad \left. - \lambda_0 c_i\frac{\log|\xi'|}{|\xi'|}\partial_i E_0(z)\right)\bigg|_+(\xi')\,\mathrm{d}\xi' \\
&\quad = \lambda_0\int_{\mathbb{R}^d\setminus\overline{B}}\left(e_1(\xi') - \left(\xi'_i + c_{ij}\frac{\xi'_j}{|\xi'|^3} + \frac{c_{ij}\xi'_j}{|\xi'|^3} + \frac{c_i}{|\xi'|^3} + \frac{c_{ijk}\xi'_j\xi'_k}{|\xi'|^5}\right)\partial_i E_0(z)\right)\mathrm{d}\xi',
\end{aligned}$$

the term of order $\mathrm{O}(1/|\xi|)$ in the far field of $e_3(\xi)$ is given by:

$$\frac{1}{4\pi}\left((\lambda_0-\lambda_*)\int\limits_B e_1(\xi')\,\mathrm{d}\xi' - \left(1-\frac{\mu_0}{\mu_*}\right)\int\limits_{\partial B}\frac{\partial e_3}{\partial \nu}\bigg|_-(\xi')\,\mathrm{d}s(\xi')\right).$$

The term

$$\int\limits_{\mathbb{R}^d\setminus\overline{B}}\left(e_1(\xi')-\xi_i'\partial_i E_0(z)-c_{ij}\frac{\xi_j'}{|\xi'|^3}\partial_i E_0(z)\right)\left(G^0(\xi,\xi')-\frac{1}{4\pi}\frac{1}{|\xi|}\right)\mathrm{d}\xi'$$

does not contribute to the $\mathrm{O}(1/|\xi|)$ terms. Thus, since

$$-\lambda_*\int\limits_B e_1(\xi')\,\mathrm{d}\xi' = \int\limits_{\partial B}\frac{\partial e_3}{\partial\nu}\bigg|_-(\xi')\,\mathrm{d}s(\xi')$$

this term is equal to:

$$\frac{\lambda_0}{4\pi}\left(1-\frac{\varepsilon_*}{\varepsilon_0}\right)\partial_i E_0(z)\int\limits_B \hat{v}_{1i}^{\mu}(\xi')\,\mathrm{d}\xi'.$$

Thus we arrive at the following behavior of $E_2$ near $z$:

$$\begin{aligned}E_2(y)\sim \lambda_0\left(\frac{\varepsilon_*}{\varepsilon_0}-1\right)&\left[\partial_i E_0(z)\left(\int\limits_B \hat{v}_{1i}^{\mu}(\xi)\,\mathrm{d}\xi\right)G^0(y,z)+E_0(z)\left(\int\limits_B \xi_k\,\mathrm{d}\xi\right)\partial_k G^0(y,z)\right]\\ &+\left(1-\frac{\mu_0}{\mu_*}\right)\left[\partial_{ij}E_0(z)\left(\int\limits_B \operatorname{grad}\hat{v}_{2ij}^{\mu,\varepsilon}(\xi)\cdot\mathbf{u}_k\,\mathrm{d}\xi\right)\partial_k G^0(y,z)\right.\\ &\left.+\,\partial_i E_0(z)\left(\int\limits_B \xi_j\partial_k\hat{v}_{1i}^{\mu}(\xi)\,\mathrm{d}\xi\right)\partial_{jk}^2 G^0(y,z)\right]\quad \text{as } y\to z.\end{aligned}\tag{4.18}$$

Combining (4.16) and (4.18) we obtain by using that

$$\lim_{\delta\to 0}\int\limits_{\partial B_\delta}\partial_{ij}^2 G^0\,\mathrm{d}s(y)=0,$$

$$\lim_{\delta\to 0}\int\limits_{\partial B_\delta}\left(E_0\frac{\partial}{\partial\nu}\partial_{jk}^2 G^0-\partial_{jk}^2 G^0\frac{\partial E_0}{\partial\nu}\right)\mathrm{d}s(y)=-\partial_{jk}^2 E_0(z),$$

$$\lim_{\delta\to 0}\int\limits_{\partial B_\delta}\frac{\partial G^0}{\partial\nu}\varphi\,\mathrm{d}s(y)=-\varphi(z),$$

$$\lim_{\delta\to 0}\int_{\partial B_\delta} \frac{\partial}{\partial\nu}\partial_j G^0\varphi\,\mathrm{d}s(y) = \partial_j\varphi(z),$$

for any smooth function $\varphi$, that the second correction $\lambda_2$ is given by:

$$\begin{aligned}\lambda_2 = \lambda_0\left(1-\frac{\varepsilon_*}{\varepsilon_0}\right)\partial_i E_0(z)\left(\int_B \left(\hat{v}_{1i}^{\mu}(\xi)+\xi_i\right)\mathrm{d}\xi\right)E_0(z)\\ -\left(1-\frac{\mu_0}{\mu_*}\right)\left[\partial_{ij}E_0(z)\left(\int_B \operatorname{grad}\hat{v}_{2ij}^{\mu,\varepsilon}(\xi)\cdot\mathbf{u}_k\,\mathrm{d}\xi\right)\partial_k E_0(z)\right.\\ \left. -\,\partial_i E_0(z)\left(\int_B \xi_j\,\partial_k\hat{v}_{1i}^{\mu}(\xi)\,\mathrm{d}\xi\right)\partial_{jk}^2 E_0(z)\right].\end{aligned} \tag{4.19}$$

If $d=2$ we can prove after simple but lengthy calculations that the behavior of $E_2$ near $z$ is given by (4.18) and therefore, formula (4.19) also holds for the second correction $\lambda_2$ in the two-dimensional case.

Finally expanding

$$\frac{1}{\omega_\alpha^2}-\frac{1}{\omega_0^2} = -\alpha^d\frac{\lambda_1}{\omega_0^4\varepsilon_0\mu_0} - \alpha^{d+1}\frac{\lambda_2}{\omega_0^4\varepsilon_0\mu_0} + \mathrm{O}\left(\alpha^{d+2}\right),$$

formulas (4.15) and (4.19) formally yield our asymptotic expansion (4.4).

### *4.2. Justification of the asymptotic expansions*

Let $X_\alpha(\Omega)$ be $L^2(\Omega)$ or $\{f\in L^2(\Omega)\colon \int_\Omega \varepsilon_\alpha f = 0\}$ equipped with the inner product $(f,g)_\alpha = \int_\Omega \varepsilon_\alpha(x) f(x) g(x)\,\mathrm{d}x$. Let the continuous linear operator $R_\alpha : X_0(\Omega)\to X_\alpha(\Omega)$ be defined by $R_\alpha(f) = \varepsilon_0 f/\varepsilon_\alpha$. Define the operator $T_\alpha : X_\alpha(\Omega)\to X_\alpha(\Omega)$ by $T_\alpha(f) = v_\alpha$ where $v_\alpha$ is the unique solution in $X_\alpha(\Omega)$ to

$$\operatorname{div}\left(\frac{1}{\mu_\alpha}\operatorname{grad} v_\alpha\right) = \varepsilon_\alpha f \quad \text{in } \Omega, \qquad v_\alpha = 0 \quad \text{or } \frac{\partial v_\alpha}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

We begin with establishing some properties of the operator $T_\alpha$.

**Lemma 4.1.** *T he following properties of the operator $T_\alpha$ hold.*

(a) *$T_\alpha : X_\alpha(\Omega)\mapsto X_\alpha(\Omega)$ is a self-adjoint compact operator.*
(b) *The family of operators $\{R_\alpha^{-1}T_\alpha R_\alpha\}_\alpha$ is collectively compact.*
(c) *The set $(-1/\omega_\alpha^2)$, where $(\omega_\alpha^2)$ is the set of resonant frequencies of the Helmholtz equation* (4.1)*, is the set of eigenvalues of $T_\alpha$.*
(d) *For any uniformly bounded sequence $f_\alpha\in X_\alpha(\Omega)$, there exists a subsequence $f_{\alpha'}$ and $f_0\in X_0(\Omega)$ such that $\|T_{\alpha'}(f_{\alpha'}) - R_{\alpha'}T_0(f_0)\|_{X_{\alpha'}(\Omega)}\to 0$ as $\alpha'\to 0$.*

(e) $\|T_\alpha(f_\alpha) - R_\alpha T_0(f_0)\|_{X_\alpha(\Omega)} \to 0$ *if* $\|f_\alpha - R_\alpha f_0\|_{X_\alpha(\Omega)} \to 0$ *for any* $f_\alpha \in X_\alpha(\Omega)$ *and* $f_0 \in X_0(\Omega)$.

**Proof.** Clearly $T_\alpha$ is a compact, self-adjoint operator from $X_\alpha(\Omega)$ to $X_\alpha(\Omega)$. From the standard $H^1$-estimates for $v_\alpha$ which are independent of $\alpha$, we see that the family of operators $\{R_\alpha^{-1} T_\alpha R_\alpha\}_\alpha$ is collectively compact. These estimates also yield (d). Now if $E_\alpha \in X_\alpha(\Omega)$ is a nonzero solution to the Helmholtz equation (4.1) then $T_\alpha(E_\alpha) = -E_\alpha/\omega_\alpha^2$ and so, $-1/\omega_\alpha^2$ is an eigenvalue of $T_\alpha$. The converse is also immediate. The proof of point (e) is slightly more delicate. Since $f_\alpha$ belongs to $X_\alpha(\Omega)$, then

$$\int_\Omega \frac{1}{\mu_\alpha} \operatorname{grad} T_\alpha(f_\alpha) \cdot \operatorname{grad} v = \int_\Omega \varepsilon_\alpha f_\alpha v, \quad \forall v \in H^1(\Omega). \tag{4.20}$$

Thus, $T_\alpha(f_\alpha)$ is bounded in $H^1(\Omega)$. A subsequence $T_{\alpha'}(f_{\alpha'})$ converges weakly to an element $g$ in $H^1(\Omega)$, this convergence is strong in $L^2(\Omega)$ due to the compactness of the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. At the limit we obtain:

$$\int_\Omega \frac{1}{\mu_0} \operatorname{grad} g \cdot \operatorname{grad} v = \int_\Omega f_0 v, \quad \forall v \in H_0^1(\Omega) \text{ or } H^1(\Omega), \tag{4.21}$$

which proves that $g = R_0 T_0(f_0)$, which does not depend on the choice of the subsequence $\alpha'$. In addition, it is clear that $R_\alpha T_0(f_0) - R_0 T_0(f_0)$ tends to 0 in $L^2(\Omega)$, which concludes the proof of (e). □

Now, direct application of classical theorems [23, Theorems 11.4 and 11.5, pp. 343–344] on spectral properties of a sequence of operators satisfying properties (a), (b), (d), and (e) that have been stated in Lemma 4.1 allow us in view of point (c) to describe in the following lemma the relative asymptotic properties and the deviation of the eigenfrequencies for the Helmholtz equation (4.1) in the presence of small imperfections to the resonant frequencies in absence of inhomogeneities.

**Lemma 4.2.** *Suppose $\omega_0$ is an eigenfrequency of multiplicity $n$ for the Helmholtz equation in absence of inhomogeneities, and let $\{E_0^i\}_{i=1}^n$ denote the corresponding eigenfunctions defined in* (4.2)–(4.3). *For $\alpha$ small enough there exist $n$ eigenfrequencies $\{\omega_\alpha^i\}_{i=1}^n$* (*counted according to multiplicity*) *for the Helmholtz equation* (4.1) *in presence of small imperfections that converge to $\omega_0$ as $\alpha$ approaches zero.*

### 4.3. *The intermediate fields $v_\alpha^i$*

For $i = 1, \ldots, n$, define $v_\alpha^i$ to be the unique solution in $X_\alpha(\Omega)$ of

$$\operatorname{div}\left(\frac{1}{\mu_\alpha} \operatorname{grad} v_\alpha^i\right) = \varepsilon_0 E_0^i \quad \text{in } \Omega,$$

$$v_\alpha^i = 0 \quad \text{or} \quad \frac{\partial v_\alpha^i}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{4.22}$$

Notice that $T_\alpha R_\alpha E_0^i = v_\alpha^i$ and the right-hand side in the first equation in (4.22) is independent of $\alpha$. Since $T_0 R_0 E_0^i = -E_0^i/\omega_0^2$ it seems that $v_\alpha^i + E_0^i/\omega_0^2$ is small in norm. To obtain our asymptotic formula, it is in fact crucial not only to prove that $v_\alpha^i + E_0^i/\omega_0^2$ is indeed small in norm but also to give an asymptotic expansion of that difference.

**Lemma 4.3.** *Let $v_\alpha^i$ be the solution to* (4.22). *Then for some constant $C$, depending on $E_0^i$ but independent of $\alpha$ and of the set of points $\{z_j\}_{j=1}^m$, the following estimate holds*:

$$\left\| v_\alpha^i + \frac{1}{\omega_0^2} E_0^i \right\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2}. \tag{4.23}$$

**Proof.** We restrict our proof to the case of a single inhomogeneity ($m = 1$). We suppose that this inhomogeneity is centered at the origin, so it is of the form $\alpha B$, with magnetic permeability $\mu_*$ and electric permittivity $\varepsilon_*$. The general case may be verified by a fairly direct iteration of the argument adding one inhomogeneity at a time. We note that by the same proof, Theorem 2.1 still holds for the case of a nonhomogeneous right-hand side. That is, if we make the change of variables $y = x/\alpha$, and let $\widetilde{\Omega} = \Omega/\alpha$, we have that

$$\left\| \operatorname{grad}_y \left( v_\alpha^i(\alpha y) + \frac{1}{\omega_0^2} E_0^i(\alpha y) + \alpha w(y) \right) \right\|_{L^2(\widetilde{\Omega})} \leqslant C\alpha^{3/2}, \tag{4.24}$$

where $w(y)$ is defined to be the unique solution to

$$\begin{cases} \Delta_y w = 0 & \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \Delta_y w = 0 & \text{in } B, \\ w & \text{is continuous} \\ & \text{across each } \partial B, \\ \displaystyle \frac{1}{\mu_0} \frac{\partial w}{\partial \nu}\bigg|_+ - \frac{1}{\mu_*} \frac{\partial w}{\partial \nu_y}\bigg|_- = -\frac{1}{\omega_0^2} \left( \frac{1}{\mu_0} - \frac{1}{\mu_*} \right) \operatorname{grad}_x E_0^i(0) \cdot \nu & \text{on } \partial B, \\ \displaystyle \lim_{|y| \to \infty} w(y) = 0. \end{cases} \tag{4.25}$$

The decay estimate at infinity for $w$ gives us, by a rescaling of the Poincaré inequality,

$$\left\| v_\alpha^i(\alpha y) + \frac{1}{\omega_0^2} E_0^i(\alpha y) + \alpha w(y) \right\|_{L^2(\widetilde{\Omega})} \leqslant C\alpha^{1/2}.$$

By changing variables back to the small domain,

$$\left\| v_\alpha^i + \frac{1}{\omega_0^2} E_0^i + \alpha w\left( \frac{x}{\alpha} \right) \right\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2}.$$

By again the decay at infinity of $w$, it is known that, in any dimension,

$$\left\| w(y) \right\|_{L^2(\widetilde{\Omega})} \leqslant C\alpha^{-1/2},$$

which rescales to

$$\|w\|_{L^2(\Omega)} \leqslant C\alpha^{d/2-1/2} \quad \text{or} \quad \|\alpha w\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2},$$

from which the result follows. □

We now proceed to construct an asymptotic expansion of $v_\alpha^i$. Let $q^*$ be the unique (scalar) solution to

$$\begin{cases} \Delta q^* = 0 & \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \Delta q^* = 1 & \text{in } B, \\ \dfrac{1}{\mu_0} \dfrac{\partial q^*}{\partial \nu}\bigg|_+ - \dfrac{1}{\mu_*} \dfrac{\partial q^*}{\partial \nu}\bigg|_- = 0 & \text{on } \partial B, \\ q^* \text{ is continuous across } \partial B, \\ \lim\limits_{|y| \to +\infty} q^*(y) = 0. \end{cases}$$

**Lemma 4.4.** *The following identity holds*:

$$\int_B \operatorname{grad} q^*(\xi) \cdot \mathbf{u}_j \,\mathrm{d}\xi = \frac{\mu_0}{\mu_0 - \mu_*} \int_B \left(\hat{v}_{1j}^\mu(\xi) - \xi_j\right) \mathrm{d}\xi.$$

**Proof.** Integrations by parts yield

$$\frac{1}{\mu_*} \int_B \left(\hat{v}_{1j}^\mu(\xi) - \xi_j\right) \mathrm{d}\xi = \frac{1}{\mu_*} \int_{\partial B} \left( \left(\hat{v}_{1j}^\mu(\xi) - \xi_j\right) \frac{\partial q^*}{\partial \nu}\bigg|_- - q^* \frac{\partial}{\partial \nu} \left(\hat{v}_{1j}^\mu(\xi) - \xi_j\right)\bigg|_- \right) \mathrm{d}s(\xi),$$

and

$$0 = \frac{1}{\mu_0} \int_{\partial B} \left( \left(\hat{v}_{1j}^\mu(\xi) - \xi_j\right) \frac{\partial q^*}{\partial \nu}\bigg|_+ - q^* \frac{\partial}{\partial \nu} \left(\hat{v}_{1j}^\mu(\xi) - \xi_j\right)\bigg|_+ \right) \mathrm{d}s(\xi).$$

Thus,

$$\frac{1}{\mu_*} \int_B \left(\hat{v}_{1j}^\mu(\xi) - \xi_j\right) \mathrm{d}\xi = \left( \frac{1}{\mu_*} - \frac{1}{\mu_0} \right) \int_{\partial B} q^*(\xi) \nu_j(\xi) \,\mathrm{d}s(\xi).$$

Since

$$\int_{\partial B} q^*(\xi) \nu_j(\xi) \,\mathrm{d}s(\xi) = \int_B \operatorname{grad} q^*(\xi) \cdot \mathbf{u}_j \,\mathrm{d}\xi,$$

we obtain that

$$\int_B \operatorname{grad} q^*(\xi) \cdot \mathbf{u}_j \,\mathrm{d}\xi = \frac{\mu_0}{\mu_0 - \mu_*} \int_B \big(\hat{v}_{1j}^{\mu}(\xi) - \xi_j\big) \,\mathrm{d}\xi,$$

as desired. □

By the same proof as Theorem 2.1 the uniform asymptotic expansion of $v_\alpha^i(x)$ holds. For $d = 2$ a similar estimate holds.

**Lemma 4.5.** *For $d = 3$ the following estimate holds*:

$$\begin{aligned}
&\bigg\| v_\alpha^i(x) + \frac{1}{\omega_0^2} E_0^i(x) \\
&\quad + \alpha \frac{1}{\omega_0^2} \partial_j E_0^i(z) \left( \hat{v}_{1j}^{\mu}\left(\frac{x-z}{\alpha}\right) - \frac{(x-z)_j}{\alpha} \right) - \alpha^2 \varepsilon_0 \left(1 - \frac{\mu_0}{\mu_*}\right) E_0^i(z) q^*\left(\frac{x-z}{\alpha}\right) \\
&\quad + \alpha^2 \frac{1}{\omega_0^2} \partial_{jk}^2 E_0^i(z) \left( v_{2jk}^{\mu,\varepsilon}\left(\frac{x-z}{\alpha}\right) - \frac{1}{2} \frac{(x-z)_j}{\alpha} \frac{(x-z)_k}{\alpha} \right) \bigg\|_{L^2(\Omega)} \leqslant C\alpha^3. \qquad (4.26)
\end{aligned}$$

**Proof** (*outline*). Consider for instance the Dirichlet problem. The proof for the Neumann problem is similar. From

$$\begin{aligned}
&\int_\Omega \frac{1}{\mu_\alpha} \operatorname{grad}\left( v_\alpha^i + \frac{1}{\omega_0^2} E_0^i \right) \cdot \operatorname{grad} v \\
&\quad = \frac{1}{\omega_0^2} \left( \frac{1}{\mu_*} - \frac{1}{\mu_0} \right) \int_{z+\alpha B} \operatorname{grad} E_0^i \cdot \operatorname{grad} v \quad \forall v \in H_0^1(\Omega),
\end{aligned}$$

and

$$\int_\Omega \frac{1}{\mu_\alpha} \operatorname{grad} q^*\left(\frac{x-z}{\alpha}\right) \cdot \operatorname{grad} v = -\frac{1}{\alpha^2} \int_{z+\alpha B} v \quad \forall v \in H_0^1(\Omega),$$

we deduce that

$$\begin{aligned}
&\int_\Omega \frac{1}{\mu_\alpha} \operatorname{grad}\left( v_\alpha^i + \frac{1}{\omega_0^2} E_0^i - \alpha^2 \varepsilon_0 \left(1 - \frac{\mu_0}{\mu_*}\right) E_0^i(z) q^*\left(\frac{x-z}{\alpha}\right) \right) \cdot \operatorname{grad} v \\
&\quad = \varepsilon_0 \left( \frac{\mu_0}{\mu_*} - 1 \right) \int_{z+\alpha B} \big(E_0^i(x) - E_0^i(z)\big) v(x) \,\mathrm{d}x \\
&\qquad + \frac{1}{\omega_0^2} \left( \frac{1}{\mu_*} - \frac{1}{\mu_0} \right) \int_{\partial(z+\alpha B)} \frac{\partial E_0^i}{\partial \nu} v \quad \forall v \in H_0^1(\Omega),
\end{aligned}$$

and therefore, the same arguments as for the proof of Theorem 2.1 can be applied to arrive at the desired uniform asymptotic expansion (4.26). □

*4.4. Proof of the asymptotic formula (4.4)*

**Lemma 4.6.** *Let $P_\alpha^i$, $1 \leqslant i \leqslant n$, denote the orthogonal projection in $X_\alpha(\Omega)$ on the eigenspace of $T_\alpha$ corresponding to the eigenvalue $-1/(\omega_\alpha^i)^2$, let $P_0$ denote the orthogonal projection in $X_0(\Omega)$ on the eigenspace of $T_0$ corresponding to the eigenvalue $-1/\omega_0^2$, and denote $P_\alpha = \sum_{i=1}^n P_\alpha^i$. We then have the following estimate relating $P_\alpha$ to $P_0$:*

$$\left\| P_\alpha R_\alpha(f) - P_0 R_0(f)\right\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2}\|f\|_{X_0(\Omega)} \tag{4.27}$$

*for all $f$ in $X_0(\Omega)$.*

**Proof.** Let $\Gamma$ be the circle in the complex plane centered at $\omega_0^2$ of radius $\delta$, $\delta$ small enough so that (4.2)–(4.3) have no other eigenvalue inside $\Gamma$. For $z \in \Gamma$, let $w_\alpha$ be the unique solution in $X_\alpha(\Omega)$ to the following equations:

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{\mu_\alpha}\operatorname{grad} w_\alpha\right) + z\varepsilon_\alpha w_\alpha = f & \text{in } \Omega, \\ w_\alpha = 0 \quad \text{or} \quad \dfrac{\partial w_\alpha}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in X_0(\Omega)$. Note that

$$(I + zT_\alpha)w_\alpha = T_\alpha\left(\frac{f}{\varepsilon_\alpha}\right).$$

Analogously, we define for any $f \in X_0(\Omega)$ and $z \in \Gamma$, $w_0$ as the unique solution in $X_0(\Omega)$ to the following equation:

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{\mu_0}\operatorname{grad} w_0\right) + z\varepsilon_\alpha w_0 = f & \text{in } \Omega, \\ w_0 = 0 \quad \text{or} \quad \dfrac{\partial w_0}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By the same proof as that of Lemma 4.3 the following estimate immediately holds,

$$\|w_\alpha - w_0\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2}\|f\|_{L^2(\Omega)}, \tag{4.28}$$

uniformly in $z \in \Gamma$.

The following calculation can be justified using classical results on symmetric compact operators as developed in [24] for example. Provided that

$$0 < \delta < \omega_0^2$$

the image of $\Gamma$ under the conformal mapping of the complex plane $z \to 1/z$ is the circle $\Gamma'$ centered at $\omega_0^2/(\omega_0^4 - \delta^2)$ and of radius $\delta/(\omega_0^4 - \delta^2)$. Note that $1/\omega_0^2$ is in the inside of $\Gamma'$, and that this mapping from $\Gamma$ to $\Gamma'$ reverses the orientation of parametrization. Setting $\zeta = 1/z$, for $\alpha$ small enough,

$$\frac{1}{2\mathrm{i}\pi}\int_\Gamma w_\alpha\,\mathrm{d}z = \frac{1}{2\mathrm{i}\pi}\int_\Gamma (I + zT_\alpha)^{-1}T_\alpha\left(\frac{f}{\varepsilon_\alpha}\right)\mathrm{d}z - \frac{1}{2\mathrm{i}\pi}\int_{\Gamma'}\frac{1}{\zeta}(\zeta I + T_\alpha)^{-1}T_\alpha\left(\frac{f}{\varepsilon_\alpha}\right)\mathrm{d}\zeta$$
$$= \sum_{i=1}^n (\omega_\alpha^i)^2 P_\alpha^i T_\alpha\left(\frac{f}{\varepsilon_\alpha}\right) = \sum_{i=1}^n (\omega_\alpha^i)^2 T_\alpha P_\alpha^i\left(\frac{f}{\varepsilon_\alpha}\right) = -P_\alpha\left(\frac{f}{\varepsilon_\alpha}\right). \tag{4.29}$$

Here we have used the fact that $T_\alpha$ and $P_\alpha^i$ commute. Similarly

$$\frac{1}{2\mathrm{i}\pi}\int_\Gamma w_0\,\mathrm{d}z = \omega_0^2 P_0 T_0\left(\frac{f}{\varepsilon_0}\right) = -P_0\left(\frac{f}{\varepsilon_0}\right). \tag{4.30}$$

Combining (4.28), (4.29), (4.30) we arrive at (4.27). □

The usual way to get to the average of the $1/(\omega_\alpha^i)^2$ is to introduce linear operators on a finite-dimensional space and use the fact that the trace is independent of the choice of a basis. Such a method is put into practice in [35] for example. We can claim in view of (4.27) that for $\alpha$ small enough $P_\alpha R_\alpha$ is one to one from $\mathbf{R}(P_0)$ into $\mathbf{R}(P_\alpha)$, where $\mathbf{R}$ denotes the range. But these two finite-dimensional spaces have the same dimension, thus $P_\alpha R_\alpha$ is an isomorphism from $\mathbf{R}(P_0)$ onto $\mathbf{R}(P_\alpha)$. Set

$$\widehat{T}_\alpha = (P_\alpha R_\alpha)^{-1} T_\alpha P_\alpha R_\alpha \quad \text{and} \quad \widehat{T}_0 = T|_{\mathbf{R}(P_0)}.$$

The trace of a finite-dimensional linear operator is independent of the choice of a basis. Consequently,

$$\frac{1}{\omega_0^2} - \frac{1}{n}\sum_{i=1}^n \frac{1}{(\omega_\alpha^i)^2} = \frac{1}{n}\mathrm{trace}\big(\widehat{T}_0 - \widehat{T}_\alpha\big) = \frac{1}{n}\sum_{i=1}^n \big\langle\big(\widehat{T}_0 - \widehat{T}_\alpha\big)E_0^i, E_0^i\big\rangle_{X_0(\Omega)}. \tag{4.31}$$

We now want to introduce $v_\alpha^i$ in $\langle \widehat{T}_\alpha E_0^i, E_0^i\rangle_{X_0(\Omega)}$. We first split the latter in two terms, the second of these two terms will be proved to be of lower order:

$$\big\langle \widehat{T}_\alpha E_0^i, E_0^i\big\rangle_{X_0(\Omega)} = \int_\Omega \varepsilon_0 (P_\alpha R_\alpha)^{-1} T_\alpha P_\alpha R_\alpha E_0^i \cdot E_0^i = \int_\Omega \varepsilon_0 R_\alpha^{-1} T_\alpha R_\alpha E_0^i \cdot E_0^i$$
$$+ \int_\Omega \varepsilon_0\big((P_\alpha R_\alpha)^{-1} T_\alpha P_\alpha R_\alpha - R_\alpha^{-1} T_\alpha R_\alpha\big) E_0^i \cdot E_0^i. \tag{4.32}$$

Recalling that $T_\alpha R_\alpha E_0^i = v_\alpha^i$ and $T_\alpha$ and $P_\alpha$ commute as linear operators of $X_\alpha(\Omega)$, we have that

$$A_\alpha = \int_\Omega \varepsilon_0\big((P_\alpha R_\alpha)^{-1} T_\alpha P_\alpha R_\alpha - R_\alpha^{-1} T_\alpha R_\alpha\big) E_0^i \cdot E_0^i$$
$$= \int_\Omega \varepsilon_0\big((P_\alpha R_\alpha)^{-1} P_\alpha - R_\alpha^{-1}\big) v_\alpha^i \cdot E_0^i.$$

Since $v_\alpha^i = \frac{1}{\varepsilon_0} R_\alpha \varepsilon_\alpha v_\alpha^i$, the term $A_\alpha$ is again equal to

$$\int_\Omega \big((P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha - I\big) \varepsilon_\alpha v_\alpha^i \cdot E_0^i.$$

But since $P_0$ is an orthogonal projection and $E_0^i$ lies in the range of $P_0$, the following identity holds

$$\int_\Omega (P_0 - I)\varepsilon_\alpha v_\alpha^i \cdot E_0^i = 0.$$

Therefore

$$A_\alpha = \int_\Omega \big((P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha - P_0\big) \varepsilon_\alpha v_\alpha^i \cdot E_0^i.$$

We use again the fact that $P_0$ is an orthogonal projection to write

$$\int_\Omega (I - P_0)(P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha \varepsilon_\alpha v_\alpha^i \cdot (P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha \varepsilon_\alpha v_\alpha^i = 0.$$

Since $P_0 = P_0 (P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha$ we rewrite $A_\alpha$ as

$$\int_\Omega \big((P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha - P_0\big) \varepsilon_\alpha v_\alpha^i \cdot \left(E_0^i + \frac{1}{\varepsilon_0}\omega_0^2 (P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha \varepsilon_\alpha v_\alpha^i\right).$$

Finally, remarking that $((P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha - P_0) E_0^i = 0$, $A_\alpha$ is equal to

$$\int_\Omega \big((P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha - P_0\big)\left(\varepsilon_\alpha v_\alpha^i + \varepsilon_0 \frac{E_0^i}{\omega_0^2}\right) \cdot \left(E_0^i + \frac{1}{\varepsilon_0}\omega_0^2 (P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha \varepsilon_\alpha v_\alpha^i\right).$$

**Lemma 4.7.** *We have*:

$$\frac{1}{\omega_0^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{(\omega_\alpha^i)^2} = -\frac{1}{n}\sum_{i=1}^{n}\int_\Omega \varepsilon_\alpha\left(v_\alpha^i + \frac{1}{\omega_0^2}E_0^i\right)E_0^i + \frac{1}{\omega_0^2}(\varepsilon_0-\varepsilon_*)\int_{z+\alpha B}\left(E_0^i\right)^2$$
$$+ \int_\Omega \left((P_\alpha R_\alpha)^{-1}P_\alpha R_\alpha - P_0\right)\left(\varepsilon_\alpha v_\alpha^i + \varepsilon_0\frac{E_0^i}{\omega_0^2}\right)$$
$$\times\left(E_0^i + \frac{1}{\varepsilon_0}\omega_0^2(P_\alpha R_\alpha)^{-1}P_\alpha R_\alpha\varepsilon_\alpha v_\alpha^i\right). \tag{4.33}$$

We notice that for all $f$ in $X_0(\Omega)$.

$$\left\|(R_\alpha - R_0)P_0(f)\right\|_{L^2(\Omega)} = \left(\int_{z+\alpha B}\left(\frac{1}{\varepsilon_*} - \frac{1}{\varepsilon_0}\right)^2\left|P_0(f)\right|^2\right)^{1/2}. \tag{4.34}$$

But all the elements in the space $\mathbf{R}(P_0)$ are smooth and any of their norm is bounded by the $L^2$ norm. We infer that

$$\left\|(R_\alpha - R_0)P_0(f)\right\|_{L^2(\Omega)} \leqslant C\alpha^d\left\|P_0(f)\right\|_{L^2(\Omega)} \leqslant C\alpha^d\|f\|_{L^2(\Omega)}.$$

Now as a consequence of (4.27) we may write, for all $f$ in $X_0(\Omega)$,

$$\left\|P_\alpha R_\alpha(f) - R_\alpha P_0(f)\right\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2}\|f\|_{X_0(\Omega)}. \tag{4.35}$$

Since the operators $P_\alpha$ are uniformly bounded in $X_\alpha(\Omega)$ and $P_\alpha^2 = P_\alpha$, it follows that

$$\left\|P_\alpha R_\alpha(f) - P_\alpha R_\alpha P_0(f)\right\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2}\|f\|_{X_0(\Omega)}. \tag{4.36}$$

Finally by definition of $(P_\alpha R_\alpha)^{-1}$ we can write the following estimate:

$$\left\|(P_\alpha R_\alpha)^{-1}P_\alpha R_\alpha(f) - P_0(f)\right\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2}\|f\|_{X_0(\Omega)}. \tag{4.37}$$

We are now ready to estimate the order of the term $A_\alpha$. First, remember that we have proved in Lemma 4.3 that

$$\left\|\varepsilon_\alpha v_\alpha^i + \varepsilon_0\frac{E_0^i}{\omega_0^2}\right\|_{X_0(\Omega)} \leqslant C\alpha^{d/2+1/2}. \tag{4.38}$$

Combining (4.37) and (4.38), we find:

$$\left\|\left((P_\alpha R_\alpha)^{-1}P_\alpha R_\alpha - P_0\right)\left(\varepsilon_\alpha v_\alpha^i + \varepsilon_0\frac{E_0^i}{\omega_0^2}\right)\right\|_{L^2(\Omega)} \leqslant C\alpha^{d+1}. \tag{4.39}$$

Next

$$
\begin{aligned}
&\left\| E_0^i + \frac{1}{\varepsilon_0}\omega_0^2 (P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha \varepsilon_\alpha v_\alpha^i \right\|_{L^2(\Omega)} \\
&\quad \leqslant \left\| P_0\left( E_0^i + \omega_0^2 \frac{\varepsilon_\alpha}{\varepsilon_0} v_\alpha^i \right) \right\|_{L^2(\Omega)} + \left\| \left( (P_\alpha R_\alpha)^{-1} P_\alpha R_\alpha - P_0 \right) \omega_0^2 \varepsilon_\alpha v_\alpha^i \right\|_{L^2(\Omega)} \\
&\quad \leqslant C \left\| E_0^i + \omega_0^2 \frac{\varepsilon_\alpha}{\varepsilon_0} v_\alpha^i \right\|_{L^2(\Omega)} + C\alpha^{d/2+1/2} \left\| \varepsilon_\alpha v_\alpha^i \right\|_{L^2(\Omega)} \leqslant C\alpha^{d/2+1/2}.
\end{aligned}
\tag{4.40}
$$

Combining (4.39) and (4.40) we find the estimate $|A_\alpha| \leqslant C\alpha^{3d/2+3/2}$. We conclude that

$$
\begin{aligned}
\frac{1}{\omega_0^2} - \frac{1}{n}\sum_{i=1}^n \frac{1}{(\omega_\alpha^i)^2} = -\frac{1}{n}\sum_{i=1}^n \int_\Omega \varepsilon_\alpha \left( v_\alpha^i + \frac{1}{\omega_0^2} E_0^i \right) E_0^i + \frac{1}{\omega_0^2}(\varepsilon_0 - \varepsilon_*) \int_{z+\alpha B} \left( E_0^i \right)^2 \\
+ \mathrm{O}\left( \alpha^{3d/2+3/2} \right).
\end{aligned}
\tag{4.41}
$$

Integrating by parts we obtain:

$$
-\frac{1}{\omega_0^2}\int_\Omega \varepsilon_0 \left( E_0^i \right)^2 + \frac{1}{\omega_0^2}\left( \frac{1}{\mu_0} - \frac{1}{\mu_*} \right) \int_{z+\alpha B} \operatorname{grad} E_0^i \cdot \operatorname{grad} v_\alpha^i = \int_\Omega \varepsilon_0 E_0^i v_\alpha^i .
$$

Thus

$$
\begin{aligned}
&-\frac{1}{n}\sum_{i=1}^n \int_\Omega \varepsilon_\alpha \left( v_\alpha^i + \frac{1}{\omega_0^2} E_0^i \right) E_0^i + \frac{1}{\omega_0^2}(\varepsilon_0 - \varepsilon_*) \int_{z+\alpha B} \left( E_0^i \right)^2 \\
&\quad = \sum_{i=1}^n -\frac{1}{\omega_0^2}\left( \frac{1}{\mu_0} - \frac{1}{\mu_*} \right) \int_{z+\alpha B} \operatorname{grad} E_0^i \cdot \operatorname{grad} v_\alpha^i + (\varepsilon_* - \varepsilon_0) \int_{z+\alpha B} E_0^i v_\alpha^i
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\frac{1}{\omega_0^2} - \frac{1}{n}\sum_{i=1}^n \frac{1}{(\omega_\alpha^i)^2} &= \sum_{i=1}^n -\frac{1}{\omega_0^2}\left( \frac{1}{\mu_0} - \frac{1}{\mu_*} \right) \int_{z+\alpha B} \operatorname{grad} E_0^i \cdot \operatorname{grad} v_\alpha^i \\
&\quad + (\varepsilon_* - \varepsilon_0) \int_{z+\alpha B} E_0^i v_\alpha^i + \mathrm{O}\left( \alpha^{3d/2+3/2} \right).
\end{aligned}
$$

Lemma 4.4 asserts that

$$
\int_{z+\alpha B} v_\alpha^i E_0^i = -\frac{1}{\omega_0^2}\alpha^d E_0^i(z)^2 - \frac{1}{\omega_0^2}\alpha^{d+1} \partial_j E_0^i(z) \left( \int_B \hat{v}_{1j}^\mu(\xi)\,\mathrm{d}\xi \right) E_0^i(z) + \mathrm{O}\left( \alpha^{d+2} \right),
$$

and

$$\int_{z+\alpha B} \operatorname{grad} v_\alpha^i \cdot \operatorname{grad} E_0^i = -\frac{1}{\omega_0^2}\alpha^d \operatorname{grad} E_0^i(z)^2 - \frac{1}{\omega_0^2}\alpha^d \operatorname{grad} E_0^i(z) M^{1,1} \operatorname{grad} E_0^i(z)$$
$$+ \alpha^{d+1}\varepsilon_0\left(1-\frac{\mu_0}{\mu_*}\right)E_0^i(z)\left(\int_B \operatorname{grad} q^*(\xi)\,\mathrm{d}\xi\right)\cdot \operatorname{grad} E_0^i(z)$$
$$-\frac{1}{\omega_0^2}\alpha^{d+1}\partial_{jk}^2 E_0^i(z) M^{2,1} \operatorname{grad} E_0^i(z).$$

Finally, Lemma 4.5 applied to (4.41) yields, in view of Lemma 4.4, our asymptotic formula (4.4).

## 5. Scattering amplitudes

Consider in this section the scattering problem in whole of $\mathbb{R}^d$, where the coefficients $\tilde{\varepsilon}_\alpha$ and $\tilde{\mu}_\alpha$ are assumed to take positive constant values $\varepsilon_e$ and $\mu_e$ outside the domain $\Omega$. Let

$$\tilde{\varepsilon}_\alpha(x) = \begin{cases} \varepsilon_e, & x \in \mathbb{R}^d \setminus \overline{\Omega},\\ \varepsilon_0, & x \in \Omega \setminus \overline{\mathcal{B}_\alpha},\\ \varepsilon_j, & x \in z_j + \alpha B_j,\ j = 1,\dots,m.\end{cases}$$

The piecewise constant electric permittivity, $\tilde{\mu}_\alpha(x)$ is defined analogously. Introduce the piecewise-constant coefficients in the absence of any inhomogeneities:

$$\tilde{\varepsilon}(x) = \begin{cases} \varepsilon_e, & x \in \mathbb{R}^d \setminus \overline{\Omega},\\ \varepsilon_0, & x \in \Omega,\end{cases} \quad \text{and} \quad \tilde{\mu}(x) = \begin{cases} \mu_e, & x \in \mathbb{R}^d \setminus \overline{\Omega},\\ \mu_0, & x \in \Omega.\end{cases}$$

Let $E_{in}(x) = \mathrm{e}^{\mathrm{i}k_e q\cdot x}$ be an incident plane wave, where $k_e = \omega\sqrt{\varepsilon_e\mu_e}$, $q \in \mathbb{R}^d$ is a unit vector giving the direction of propagation.

Let $E_\alpha$ be the unique solution to the Helmholtz equation,

$$\operatorname{div} \frac{1}{\tilde{\mu}_\alpha} \operatorname{grad} E_\alpha + \omega^2 \tilde{\varepsilon}_\alpha E_\alpha = 0 \quad \text{in } \mathbb{R}^d, \tag{5.1}$$

subject to the radiation condition as $|x| \to +\infty$:

$$\left|\frac{\partial}{\partial|x|}(E_\alpha - E_{in}) - \mathrm{i}k_e(E_\alpha - E_{in})\right| = \mathrm{O}\left(\frac{1}{|x|^{d-1}}\right). \tag{5.2}$$

The electric field $E_0$ in the absence of any inhomogeneities satisfies,

$$\operatorname{div} \frac{1}{\tilde{\mu}} \operatorname{grad} E_0 + \omega^2 \tilde{\varepsilon} E_0 = 0 \quad \text{in } \mathbb{R}^d, \tag{5.3}$$

with the radiation condition as $|x| \to +\infty$:

$$\left|\frac{\partial}{\partial|x|}(E_0 - E_{in}) - \mathrm{i}k_e(E_0 - E_{in})\right| = \mathrm{O}\left(\frac{1}{|x|^{d-1}}\right).$$

The scattering amplitude, $A_\alpha(x/|x|, q, \omega)$, is defined to be the function satisfying

$$E_\alpha(x) = E_{in}(x) + \frac{\mathrm{e}^{\mathrm{i}k_e|x|}}{|x|^{(d-1)/2}} A_\alpha\left(\frac{x}{|x|}, q, \omega\right) + \mathrm{O}\left(\frac{1}{|x|^{(d+1)/2}}\right)$$

as $|x| \to +\infty$.

Introduce the Green's function $\widetilde{G}$ of the Helmholtz equation:

$$\operatorname{div}_y \frac{1}{\tilde{\mu}(y)} \operatorname{grad}_y \widetilde{G}(x, y) + \omega^2 \tilde{\varepsilon}(y) \widetilde{G}(x, y) = -\delta_x \quad \text{in } \mathbb{R}^d, \tag{5.4}$$

subject to the radiation condition as $|y| \to +\infty$:

$$\left|\frac{\partial}{\partial|y|}\widetilde{G} - \mathrm{i}k_e \widetilde{G}\right| = \mathrm{O}\left(\frac{1}{|y|^{d-1}}\right). \tag{5.5}$$

The existence and uniqueness of a solution to (5.4)–(5.5) can be proved by using the Phillips' device [22].

In this section we find and prove a formula, asymptotic with respect to the inhomogeneity size $\alpha$, for $A_\alpha$ in terms of $A_0$, that is the scattering amplitude in the absence of any inhomogeneities. The leading-order term in this formula was derived in [5].

### *5.1. Formal derivations*

Let us first proceed *formally* to derive such an asymptotic formula. We begin by establishing a Lippmann–Schwinger representation formula for $E_\alpha$. Let $x \in \mathbb{R}^d$ be an arbitrary point and choose an open ball $\mathcal{B}$, with exterior unit normal $\nu$, containing the domain $\Omega$ such that $x \in \mathcal{B}$. From Green's formula applied to $E_\alpha$ in $\mathcal{B}$, we have:

$$\begin{aligned} E_\alpha(x) = {} & \frac{1}{\mu_e} \int_{\partial\mathcal{B}} \left(\frac{\partial E_\alpha}{\partial\nu}(y)\widetilde{G}(x, y) - E_\alpha(y)\frac{\partial \widetilde{G}}{\partial\nu(y)}(x, y)\right) \mathrm{d}s(y) \\ & + \omega^2 \sum_{j=1}^m (\varepsilon_j - \varepsilon_0) \int_{z_j+\alpha B_j} \widetilde{G}(x, y) E_\alpha(y)\, \mathrm{d}y \\ & + \sum_{j=1}^m \left(\frac{1}{\mu_0} - \frac{1}{\mu_j}\right) \int_{z_j+\alpha B_j} \operatorname{grad}_y \widetilde{G}(x, y) \cdot \operatorname{grad} E_\alpha(y)\, \mathrm{d}y. \end{aligned} \tag{5.6}$$

Green's formula applied to $E_0$ in $\mathcal{B}$ gives:

$$E_0(x) = \frac{1}{\mu_e} \int_{\partial\mathcal{B}} \left( \frac{\partial E_0}{\partial \nu}(y) \widetilde{G}(x, y) - E_0(y) \frac{\partial \widetilde{G}}{\partial \nu(y)}(x, y) \right) \mathrm{d}s(y). \tag{5.7}$$

Finally, from the Green's formula, applied to $E_\alpha - E_0$ in $\mathbb{R}^d \setminus \overline{\mathcal{B}}$, and the radiation condition we see that

$$\int_{\partial\mathcal{B}} \left( \frac{\partial(E_\alpha - E_0)}{\partial \nu}(y) \widetilde{G}(x, y) - (E_\alpha - E_0)(y) \frac{\partial \widetilde{G}}{\partial \nu(y)}(x, y) \right) \mathrm{d}s(y) = 0. \tag{5.8}$$

Combining (5.6)–(5.8) we conclude that

$$\begin{aligned} E_\alpha(x) = E_0(x) &+ \omega^2 \sum_{j=1}^m (\varepsilon_j - \varepsilon_0) \int_{z_j + \alpha B_j} \widetilde{G}(x, y) E_\alpha(y)\, \mathrm{d}y \\ &+ \sum_{j=1}^m \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \int_{z_j + \alpha B_j} \operatorname{grad}_y \widetilde{G}(x, y) \cdot \operatorname{grad} E_\alpha(y)\, \mathrm{d}y. \end{aligned} \tag{5.9}$$

Therefore

$$\begin{aligned} A_\alpha\left( \frac{x}{|x|}, q, \omega \right) = A_0\left( \frac{x}{|x|}, q, \omega \right) &+ \omega^2 \sum_{j=1}^m (\varepsilon_j - \varepsilon_0) \int_{z_j + \alpha B_j} \widetilde{G}_\infty\left( \frac{x}{|x|}, y, \omega \right) E_\alpha(y)\, \mathrm{d}y \\ &+ \sum_{j=1}^m \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \int_{z_j + \alpha B_j} \operatorname{grad}_y \widetilde{G}_\infty\left( \frac{x}{|x|}, y, \omega \right) \cdot \operatorname{grad} E_\alpha(y)\, \mathrm{d}y, \end{aligned} \tag{5.10}$$

where $\widetilde{G}_\infty(x|x|, y, \omega)$ is defined to be the function which satisfies:

$$\begin{aligned} \widetilde{G}(x, y) = \frac{\mathrm{e}^{\mathrm{i}k_e|x|}}{|x|^{(d-1)/2}} \widetilde{G}_\infty\left( \frac{x}{|x|}, y, \omega \right) \\ + \mathrm{O}\left( \frac{1}{|x|^{(d+1)/2}} \right) \quad \text{as } |x| \to +\infty \text{ and } y \text{ is fixed.} \end{aligned} \tag{5.11}$$

Using the inner approximation $E_\alpha|_{z_j + \alpha B_j} = E_0(z_j) + \alpha \partial_i E_0(z) \hat{v}_{1i}^\mu((x - z_j)/\alpha) + \mathrm{o}(\alpha)$ from Section 2 we formally write that

$$
\begin{aligned}
&\int\limits_{z_j+\alpha B_j} \widetilde{G}_\infty\left(\frac{x}{|x|}, y, \omega\right) E_\alpha(y)\,\mathrm{d}y \\
&\quad = \alpha^d \widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right) E_0(z_j)|B_j| + \alpha^{d+1}\left[\left(\int\limits_B \xi_q\,\mathrm{d}\xi\right)\partial_q \widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right) E_0(z_j)\right. \\
&\qquad \left. + \left(\int\limits_{B_j} \hat{v}^\mu_{1q}(\xi)\,\mathrm{d}\xi\right)\partial_q E_0(z_j)\widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right] + \mathrm{o}\big(\alpha^{d+1}\big),
\end{aligned} \tag{5.12}
$$

and

$$
\begin{aligned}
&\int\limits_{z_j+\alpha B_j} \operatorname{grad}_y \widetilde{G}_\infty\left(\frac{x}{|x|}, y, \omega\right)\cdot \operatorname{grad} E_\alpha(y)\,\mathrm{d}y \\
&\quad = \alpha^d\left[M_j^{1,1}\left(\frac{\mu_j}{\mu_0}\right)\operatorname{grad}_y \widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right]\cdot \operatorname{grad} E_0(z_j) \\
&\qquad + \alpha^{d+1}\sum_{j=1}^m \left(1-\frac{\mu_0}{\mu_j}\right)\left[\partial^2_{np}E_0(z_j)\big(M_j^{2,1}\big)_{npq}\partial_q \widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right. \\
&\qquad\qquad \left. + \partial_n E_0(z_j)\big(M_j^{1,2}\big)_{npq}\partial^2_{pq}\widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right] \\
&\qquad + \mathrm{o}\big(\alpha^{d+1}\big),
\end{aligned} \tag{5.13}
$$

where $M_j^{1,1}, M_j^{1,2}, M_j^{2,1}$ are the polarization tensors of the domains $B_j$.

Inserting these formal asymptotic expansions into (5.10) we arrive at the following asymptotic expansion for the scattering amplitude $A_\alpha$:

$$
\begin{aligned}
&A_\alpha\left(\frac{x}{|x|}, q, \omega\right) \\
&\quad = A_0\left(\frac{x}{|x|}, q, \omega\right) + \alpha^d \sum_{j=1}^m \left[\omega^2(\varepsilon_j-\varepsilon_0)\widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)E_0(z_j)\right. \\
&\qquad\qquad + \left(\frac{1}{\mu_0}-\frac{1}{\mu_j}\right)\left(M_j^{1,1}\left(\frac{\mu_j}{\mu_0}\right)\operatorname{grad}_y \widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right) \\
&\qquad\qquad \left. \times \operatorname{grad} E_0(z_j)\right] \\
&\qquad + \alpha^{d+1}\sum_{j=1}^m\left[\left(\frac{1}{\mu_0}-\frac{1}{\mu_j}\right)\left(1-\frac{\mu_0}{\mu_j}\right)\left(\partial^2_{np}E_0(z_j)\big(M_j^{2,1}\big)_{npq}\partial_q\widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right.\right. \\
&\qquad\qquad \left. + \partial_n E_0(z_j)\big(M_j^{1,2}\big)_{npq}\partial^2_{pq}\widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right)
\end{aligned}
$$

$$+\,\omega^2(\varepsilon_j-\varepsilon_0)\left(\partial_n E_0(z_j)\left(\int_{B_j}\hat{v}_{1q}^{\mu}(\xi)\,\mathrm{d}\xi\right)\widetilde{G}_\infty\left(\frac{x}{|x|},z_j,\omega\right)\right.$$
$$\left.\left.+\,E_0(z_j)\left(\int_{B_j}\xi_n\,\mathrm{d}\xi\right)\partial_n\widetilde{G}_\infty\left(\frac{x}{|x|},z_j,\omega\right)\right)\right]$$
$$+\,\mathrm{o}\big(\alpha^{d+1}\big). \tag{5.14}$$

## 5.2. *Proof of the asymptotic formula (5.14)*

In order to establish rigorously the asymptotic formula (5.14) for the scattering amplitude it suffices to prove that (5.12) and (5.13) hold.

Consider the equation for $E_\alpha$ in the *exterior* of $\Omega$, multiply by the Green's function $\widetilde{G}$ and integrate by parts to get that, for $x\in\mathbb{R}^d\setminus\overline{\Omega}$:

$$E_\alpha(x)=E_0(x)+\frac{1}{\mu_e}\int_{\partial\Omega}\left(\frac{\partial E_\alpha}{\partial\nu}\bigg|_+(y)\widetilde{G}(x,y)-E_\alpha(y)\frac{\partial\widetilde{G}}{\partial\nu(y)}\bigg|_+(x,y)\right)\mathrm{d}s(y),$$

where $\nu$ is the outward unit normal to $\partial\Omega$. The jump condition for $\partial E_\alpha/\partial\nu$ on $\partial\Omega$ yields

$$E_\alpha(x)=E_0(x)-\frac{1}{\mu_e}\int_{\partial\Omega}\frac{\partial\widetilde{G}}{\partial\nu(y)}\bigg|_+(x,y)E_\alpha(y)\,\mathrm{d}s(y)$$
$$+\,\frac{1}{\mu_0}\int_{\partial\Omega}\widetilde{G}(x,y)\frac{\partial E_\alpha}{\partial\nu}\bigg|_-(y)\,\mathrm{d}s(y).$$

Of course, the above equations does not hold up to the boundary of $\Omega$, but if we take the limit as $x\to\partial\Omega$, we get from for instance [17,34] that

$$\frac{1}{2}E_\alpha|_{\partial\Omega}(x)=E_0(x)|_{\partial\Omega}-\frac{1}{\mu_e}\int_{\partial\Omega}\frac{\partial\widetilde{G}}{\partial\nu(y)}\bigg|_+(x,y)E_\alpha(y)\,\mathrm{d}s(y)$$
$$+\,\frac{1}{\mu_0}\int_{\partial\Omega}\widetilde{G}(x,y)\frac{\partial E_\alpha}{\partial\nu}\bigg|_-(y)\,\mathrm{d}s(y), \tag{5.15}$$

for $x\in\partial\Omega$.

Now define the following *Dirichlet to Neumann map*:

$$N_\alpha: H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega),\qquad N_\alpha(f)=\frac{\partial v_\alpha}{\partial\nu},$$

where $v_\alpha$ is the solution to

$$\begin{aligned}&\left(\operatorname{div}\frac{1}{\tilde{\mu}_\alpha}\operatorname{grad}+\omega^2\tilde{\varepsilon}_\alpha\right)v_\alpha=0 \quad \text{in } \Omega,\\ &v_\alpha=f \quad \text{on } \partial\Omega.\end{aligned} \tag{5.16}$$

Hence,

$$N_\alpha(E_\alpha|_{\partial\Omega})=\left.\frac{\partial E_\alpha}{\partial\nu}\right|_-.$$

Similarly, let

$$N_0: H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)$$

be the Neumann to Dirichlet map for the limiting problem, so that

$$N_0(E_0|_{\partial\Omega})=\left.\frac{\partial E_0}{\partial\nu}\right|_-.$$

We also define the single and double layer potential operators:

$$S: H^{-1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega) \quad \text{and} \quad D: H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega),$$

where

$$S: g\to\int_{\partial\Omega}\widetilde{G}(x,y)g(y)\,\mathrm{d}s(y) \quad \text{and} \quad D: f\to\int_{\partial\Omega}\left.\frac{\partial\widetilde{G}}{\partial\nu(y)}\right|_+(x,y)f(y)\,\mathrm{d}s(y).$$

Using this operator notation, we readily see from (5.15) that we have:

$$\left(\frac{I}{2}+\frac{1}{\mu_e}D-\frac{1}{\mu_0}SN_\alpha\right)(E_\alpha|_{\partial\Omega})=E_0|_{\partial\Omega}.$$

Similarly, $E_0$ satisfies:

$$\left(\frac{I}{2}+\frac{1}{\mu_e}D-\frac{1}{\mu_0}SN_0\right)(E_0|_{\partial\Omega})=E_0|_{\partial\Omega}.$$

Define:

$$T_\alpha(\omega): H^{1/2}(\partial\Omega)\to H^{1/2}(\partial\Omega)$$

by

$$T_\alpha(\omega)=\frac{I}{2}+\frac{1}{\mu_e}D-\frac{1}{\mu_0}SN_\alpha \tag{5.17}$$

and let

$$T_0(\omega) = \frac{I}{2} + \frac{1}{\mu_e} D - \frac{1}{\mu_0} S N_0. \tag{5.18}$$

By subtracting the two above equations we have that

$$T_\alpha(\omega)(E_\alpha|_{\partial\Omega}) - T_0(\omega)(E_0|_{\partial\Omega}) = 0,$$

and hence

$$T_\alpha(\omega)\big((E_\alpha - E_0)|_{\partial\Omega}\big) = \frac{1}{\mu_0} S(N_0 - N_\alpha)(E_0|_{\partial\Omega}).$$

We will need the following lemma:

**Lemma 5.1.** *Let $T_\alpha$ be defined by* (5.17) *and $T_0$ by* (5.18). *Then we have the following properties*:

(a) *$T_\alpha$ converges to $T_0$ pointwise.*
(b) *The family of operators $\{T_\alpha - T_0\}_\alpha$ is collectively compact.*
(c) *There exists a constant $C$ that is independent of $\alpha$ and the set of points $\{z_j\}_{j=1}^m$ such that for any $f \in H^{1/2}(\partial\Omega)$, $T_\alpha^{-1}$ exists and*

$$\big\|T_\alpha^{-1} f\big\|_{H^{1/2}(\partial\Omega)} \leqslant C\|f\|_{H^{1/2}(\partial\Omega)}.$$

(d) *Let $f \in H^{1/2}(\partial\Omega)$ and $u$ denote the solution to $(\Delta + k^2)u = 0$ in $\Omega$ and $u = f$ on $\partial\Omega$. The following asymptotic formula holds*:

$$\begin{aligned}
&(T_0 - T_\alpha)(f)(x) \\
&\quad = \frac{1}{\mu_0} S(N_0 - N_\alpha)(f)(x) \\
&\quad = \frac{1}{\mu_0}\alpha^d \sum_{j=1}^m \bigg[\omega^2(\varepsilon_j - \varepsilon_0)\widetilde{G}(x, z_j)u(z_j) \\
&\qquad\qquad + \left(\frac{1}{\mu_0} - \frac{1}{\mu_j}\right)\left(M_j^{1,1}\left(\frac{\mu_j}{\mu_0}\right) \operatorname{grad}_y \widetilde{G}(x, z_j)\right) \cdot \operatorname{grad} u(z_j)\bigg] \\
&\qquad + \alpha^{d+1} \sum_{j=1}^m \bigg[\left(\frac{1}{\mu_0} - \frac{1}{\mu_j}\right)\left(1 - \frac{\mu_0}{\mu_j}\right)\Big(\partial_{np}^2 u(z_j)\big(M_j^{2,1}\big)_{npq}\partial_q \widetilde{G}(x, z_j) \\
&\qquad\qquad + \partial_n u(z_j)\big(M_j^{1,2}\big)_{npq}\partial_{pq}^2 \widetilde{G}(x, z_j)\Big)
\end{aligned}$$

$$+\omega^2(\varepsilon_j-\varepsilon_0)\bigg(\partial_n u(z_j)\bigg(\int_{B_j}\hat{v}_{1q}^{\mu}(\xi)\,\mathrm{d}\xi\bigg)\widetilde{G}(x,z_j) + u(z_j)\bigg(\int_{B_j}\xi_n\,\mathrm{d}\xi\bigg)\partial_n\widetilde{G}(x,z_j)\bigg)\bigg] + \mathrm{o}\big(\alpha^{d+1}\big),$$

*where the asymptotic term* $\mathrm{o}(\alpha^{d+1})$ *is independent of* $x\in\partial\Omega$ *and the set of points* $\{z_j\}_{j=1}^m$.

**Proof.** Define $\widetilde{G}_D(x,y)$ to be the Dirichlet Green's function for $\Omega$,

$$\begin{aligned}&\Delta_y\widetilde{G}_D(x,y)+k^2\widetilde{G}_D(x,y)=\delta_x \quad \text{in } \Omega,\\ &\widetilde{G}_D(x,y)=0 \quad \text{on } \partial\Omega.\end{aligned}\tag{5.19}$$

Recall that

$$N_\alpha f-N_0 f=\frac{\partial u_\alpha}{\partial\nu}-\frac{\partial u_0}{\partial\nu},$$

where

$$\begin{aligned}&\operatorname{div}\frac{1}{\mu_\alpha}\operatorname{grad}u_\alpha+\omega^2\varepsilon_\alpha u_\alpha=0 \quad \text{in } \Omega,\\ &u_\alpha=f \quad \text{on } \partial\Omega,\end{aligned}\tag{5.20}$$

and

$$\begin{aligned}&\operatorname{div}\frac{1}{\mu_0}\operatorname{grad}u_0+\omega^2\varepsilon_0 u_0=0 \quad \text{in } \Omega,\\ &u_0=f \quad \text{on } \partial\Omega.\end{aligned}\tag{5.21}$$

Let $u_\alpha$ and $u=u_0$ be defined as above. Then we have the pointwise expansion:

$$\begin{aligned}(N_\alpha-N_0)(f) &= \frac{\partial u_\alpha}{\partial\nu}(x)-\frac{\partial u}{\partial\nu}(x)\\ &=\alpha^d\sum_{j=1}^m\bigg[\omega^2(\varepsilon_j-\varepsilon_0)\frac{\partial}{\partial\nu(x)}\widetilde{G}_D(x,z_j)u(z_j)\\ &\quad+\bigg(\frac{1}{\mu_0}-\frac{1}{\mu_j}\bigg)\bigg(M_j^{1,1}\bigg(\frac{\mu_j}{\mu_0}\bigg)\operatorname{grad}_y\frac{\partial}{\partial\nu(x)}\widetilde{G}_D(x,z_j)\bigg)\cdot\operatorname{grad}u(z_j)\bigg]\end{aligned}$$

$$
\begin{aligned}
&+ \alpha^{d+1} \sum_{j=1}^{m} \Bigg[ \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \left( 1 - \frac{\mu_0}{\mu_j} \right) \Bigg( \partial^2_{np} u(z_j) \big( M_j^{2,1} \big)_{npq} \partial_q \frac{\partial}{\partial \nu(x)} \widetilde{G}_D(x, z_j) \\
&\qquad\qquad + \partial_n u(z_j) \big( M_j^{1,2} \big)_{npq} \partial^2_{pq} \frac{\partial}{\partial \nu(x)} \widetilde{G}_D(x, z_j) \Bigg) \\
&\qquad + \omega^2 (\varepsilon_j - \varepsilon_0) \Bigg( \partial_n u(z_j) \bigg( \int_{B_j} \hat{v}^{\mu}_{1q}(\xi) \, \mathrm{d}\xi \bigg) \frac{\partial}{\partial \nu(x)} \widetilde{G}_D(x, z_j) \\
&\qquad\qquad + u(z_j) \bigg( \int_{B_j} \xi_n \, \mathrm{d}\xi \bigg) \partial_n \frac{\partial}{\partial \nu(x)} \widetilde{G}_D(x, z_j) \Bigg) \Bigg] \\
&+ \mathrm{o}\big(\alpha^{d+1}\big),
\end{aligned} \tag{5.22}
$$

where the term $\mathrm{o}(\alpha^{d+1})$ is uniform for $x \in \partial\Omega$.

We are now ready to prove Lemma 5.1. Integration by parts yields

$$
\int_{\partial\Omega} \widetilde{G}(x, y) \frac{\partial}{\partial \nu_y} \big( \mathrm{grad}_z \, \widetilde{G}_D(y, z) \big) \, \mathrm{d}\sigma_y = \mathrm{grad}_z \, \widetilde{G}(x, z) \tag{5.23}
$$

and

$$
\int_{\partial\Omega} \widetilde{G}(x, y) \frac{\partial}{\partial \nu_y} \big( \widetilde{G}_D(y, z) \big) \, \mathrm{d}\sigma_y = \widetilde{G}(x, z). \tag{5.24}
$$

By applying the operator $S$ to (5.22) and using (5.23) we arrive at the promised asymptotic expansion (5.19) , which along with the fact that the operator $S$ is bounded implies that $T_\alpha$ converges to $T_0$ pointwise which is the claim in point (a). Furthermore, since the points $z_j$ are away from the boundary $\partial\Omega$, it follows from (5.19) that the family of operators $T_\alpha - T_0$ is collectively compact and so, point (b) holds. Rewriting $T_\alpha = T_0 + (T_\alpha - T_0)$ and recalling that the operator $T_0$ is invertible, it follows immediately that $T_\alpha^{-1}$ is well defined and point (c) in Lemma 5.1 holds. □

Define the corrections

$$
\begin{aligned}
E_1(x) = \sum_{j=1}^{m} \Bigg[ & \omega^2 (\varepsilon_j - \varepsilon_0) \widetilde{G}(x, z_j) E_0(z_j) \\
& + \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \left( M_j^{1,1} \left( \frac{\mu_j}{\mu_0} \right) \mathrm{grad}_y \, \widetilde{G}(x, z_j) \right) \cdot \mathrm{grad} \, E_0(z_j) \Bigg],
\end{aligned} \tag{5.25}
$$

and

$$E_2(x) = \sum_{j=1}^{m} \bigg[ \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \left( 1 - \frac{\mu_0}{\mu_j} \right) \Big( \partial^2_{np} E_0(z_j) \big( M_j^{2,1} \big)_{npq} \partial_q \widetilde{G}(x, z_j) + \partial_n E_0(z_j) \big( M_j^{1,2} \big)_{npq} \partial^2_{pq} \widetilde{G}(x, z_j) \Big) + \omega^2 (\varepsilon_j - \varepsilon_0) \bigg( \partial_n E_0(z_j) \bigg( \int_{B_j} \hat{v}^{\mu}_{1q}(\xi)\, \mathrm{d}\xi \bigg) \widetilde{G}(x, z_j) + E_0(z_j) \bigg( \int_{B_j} \xi_n \, \mathrm{d}\xi \bigg) \partial_n \widetilde{G}(x, z_j) \bigg) \bigg] \tag{5.26}$$

for $x \neq z_j, \forall j = 1, \ldots, m$.

We have therefore shown that

$$T_\alpha \big( (E_\alpha - E_0)|_{\partial\Omega} \big) = \alpha^d E_1|_{\partial\Omega} + \alpha^{d+1} E_2|_{\partial\Omega} + \mathrm{o}\big(\alpha^{d+1}\big), \tag{5.27}$$

uniformly for $x \in \partial\Omega$. Note that from the definition of the Green function $\widetilde{G}$, $E_1$ satisfies:

$$\big(\Delta + k^2\big) E_1 = \sum_{j=1}^{m} \bigg[ \omega^2 (\varepsilon_j - \varepsilon_0) E_0(z_j) \delta_{z_j} + \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \left( M_j^{1,1} \left( \frac{\mu_j}{\mu_0} \right) \operatorname{grad} \delta_{z_j} \right) \cdot \operatorname{grad} E_0(z_j) \bigg], \tag{5.28}$$

in the sense of distributions, where $\delta_{z_j}$ is the Dirac delta function at the point $z_j$. The second-order correction term $E_2$ satisfies:

$$\big(\Delta + k^2\big) E_2 = \sum_{j=1}^{m} \bigg[ \left( \frac{1}{\mu_0} - \frac{1}{\mu_j} \right) \left( 1 - \frac{\mu_0}{\mu_j} \right) \Big( \partial^2_{np} E_0(z_j) \big( M_j^{2,1} \big)_{npq} \partial_q \delta_{z_j} + \partial_n E_0(z_j) \big( M_j^{1,2} \big)_{npq} \partial^2_{pq} \delta_{z_j} \Big) + \omega^2 (\varepsilon_j - \varepsilon_0) \bigg( \partial_n E_0(z_j) \bigg( \int_{B_j} \hat{v}^{\mu}_{1q}(\xi)\, \mathrm{d}\xi \bigg) \delta_{z_j} + E_0(z_j) \bigg( \int_{B_j} \xi_n \, \mathrm{d}\xi \bigg) \partial_n \delta_{z_j} \bigg) \bigg]. \tag{5.29}$$

**Lemma 5.2.** *Let the correction term $E_i$, $i = 1, 2$, be defined by* (5.25)–(5.26). *Then we have*:

$$T_0 (E_i|_{\partial\Omega}) = E_i|_{\partial\Omega}, \quad i = 1, 2.$$

**Proof.** Multiplying (5.28) and (5.29) by $\widetilde{G}$, integrating by parts over $\Omega$ and taking the limit as $x \to \partial\Omega$, we get:

$$\frac{1}{2}E_i|_{\partial\Omega} - \int_{\partial\Omega} \frac{\partial \widetilde{G}}{\partial \nu} E_i(y)\,\mathrm{d}s(y) + \int_{\partial\Omega} \widetilde{G} \frac{\partial E_i}{\partial \nu}(y)\,\mathrm{d}s(y) = 0,$$

for $x \in \partial\Omega$, $i = 1, 2$.

Define $v_i$, $i = 1, 2$, as the unique solution to

$$\begin{cases} \Delta v_i + k^2 v_i = 0 & \text{in } \Omega, \\ v_i = E_i & \text{on } \partial\Omega, \end{cases}$$

that is,

$$\frac{\partial v_i}{\partial \nu} = N_0(E_i|_{\partial\Omega}).$$

Green's formula yields for any $x \in \Omega$ away from the centers of the inhomogeneities,

$$\int_{\partial\Omega} \widetilde{G}(x, y) \frac{\partial}{\partial \nu}(E_i - v_i)(y)\,\mathrm{d}s(y) = v_i(x).$$

Hence for $x \in \partial\Omega$,

$$\int_{\partial\Omega} \widetilde{G}(x, y) \frac{\partial}{\partial \nu}(E_i - v_i)(y)\,\mathrm{d}s(y) = E_i(x).$$

Using this we can rewrite:

$$\begin{aligned} \int_{\partial\Omega} \widetilde{G} \frac{\partial E_i}{\partial \nu}(y)\,\mathrm{d}s(y) &= \int_{\partial\Omega} \widetilde{G} N_0(E_i)(y)\,\mathrm{d}s(y) + \int_{\partial\Omega} \widetilde{G}\left(\frac{\partial E_i}{\partial \nu}(y) - N_0(E_i)(y)\right)\mathrm{d}s(y) \\ &= \int_{\partial\Omega} \widetilde{G} N_0(E_i)(y)\,\mathrm{d}s(y) + E_i(x), \end{aligned}$$

from which it follows that

$$\frac{1}{2}E_i|_{\partial\Omega} - \int_{\partial\Omega} \frac{\partial \widetilde{G}}{\partial \nu} E_i(y)\,\mathrm{d}s(y) + \int_{\partial\Omega} \widetilde{G} N_0(E_i)(y)\,\mathrm{d}s(y) = E_i(x),$$

for $x \in \partial\Omega$. This just says exactly that $T_0(E_i|_{\partial\Omega}) = E_i|_{\partial\Omega}$, $i = 1, 2$. □

**Lemma 5.3.** *The following estimate holds*:

$$\left\| E_\alpha - E_0 - \alpha^d E_1 - \alpha^{d+1} E_2 \right\|_{H^{1/2}(\partial\Omega)} = \mathrm{o}\big(\alpha^{d+1}\big), \tag{5.30}$$

*where the term* $\mathrm{o}(\alpha^{d+1})$ *goes to zero faster than* $\alpha^{d+1}$ *independent of the set of points* $\{z_j\}_{j=1}^m$.

**Proof.** From (5.27) it follows that

$$\begin{aligned} &T_\alpha\big(\big(E_\alpha - E_0 - \alpha^d E_1 - \alpha^{d+1} E_2\big)\big|_{\partial\Omega}\big) \\ &\quad = \alpha^d E_1|_{\partial\Omega} - \alpha^{d+1} E_2|_{\partial\Omega} - \alpha^d T_\alpha(E_1|_{\partial\Omega}) - \alpha^{d+1} T_\alpha(E_2|_{\partial\Omega}) + \mathrm{o}\big(\alpha^{d+1}\big). \end{aligned}$$

Lemma 5.2 yields:

$$\begin{aligned} &T_\alpha\big(\big(E_\alpha - E_0 - \alpha^d E_1 - \alpha^{d+1} E_2\big)\big|_{\partial\Omega}\big) \\ &\quad = \alpha^d (T_0 - T_\alpha)(E_1|_{\partial\Omega}) + \alpha^{d+1}(T_0 - T_\alpha)(E_2|_{\partial\Omega}) + \mathrm{o}\big(\alpha^{d+1}\big). \end{aligned}$$

Therefore, due to the pointwise convergence of $T_\alpha$ to $T_0$, we obtain:

$$T_\alpha\big(\big(E_\alpha - E_0 - \alpha^d E_1 - \alpha^{d+1} E_2\big)\big|_{\partial\Omega}\big) = \mathrm{o}\big(\alpha^{d+1}\big),$$

which leads, by using point (c) in Lemma 5.1, to the desired estimate (5.30). □

From this lemma we obtain the following theorem:

**Theorem 5.1.** *Let* $E_\alpha$ *be the solution to the scattering problem* (5.1)–(5.2) *and let* $M_j^{1,1}$, $M_j^{1,2}$, *and* $M_j^{2,1}$ *be the polarization tensors of order* $(1,1), (1,2)$, *and* $(2,1)$ *for the shapes* $B_j$. *Then for* $x \in \mathbb{R}^d \setminus \overline{\Omega}$ *bounded away from* $\partial\Omega$ *we have the pointwise expansion*:

$$\begin{aligned} &E_\alpha(x) \\ &\quad = E_0(x) + \alpha^d \sum_{j=1}^m \bigg[ \omega^2(\varepsilon_j - \varepsilon_0)\widetilde{G}(x, z_j) E_0(z_j) \\ &\qquad\qquad + \left(\frac{1}{\mu_0} - \frac{1}{\mu_j}\right)\left(M_j^{1,1}\left(\frac{\mu_j}{\mu_0}\right) \operatorname{grad}_y \widetilde{G}(x, z_j)\right) \cdot \operatorname{grad} E_0(z_j)\bigg] \\ &\qquad + \alpha^{d+1} \sum_{j=1}^m \bigg[\left(\frac{1}{\mu_0} - \frac{1}{\mu_j}\right)\left(1 - \frac{\mu_0}{\mu_j}\right)\big(\partial^2_{np} E_0(z_j)\big(M_j^{2,1}\big)_{npq} \partial_q \widetilde{G}(x, z_j) \\ &\qquad\qquad + \partial_n E_0(z_j)\big(M_j^{1,2}\big)_{npq} \partial^2_{pq} \widetilde{G}(x, z_j)\big) \\ &\qquad\qquad + \omega^4(\varepsilon_j - \varepsilon_0)\left(\partial_n E_0(z_j)\left(\int_{B_j} \hat{v}^\mu_{1q}(\xi)\,\mathrm{d}\xi\right)\widetilde{G}(x, z_j)\right) \end{aligned}$$

$$+ E_0(z_j)\left(\int_{B_j} \xi_n \,\mathrm{d}\xi\right)\partial_n \widetilde{G}(x, z_j)\bigg)\bigg] + \mathrm{o}\big(\alpha^{d+1}\big). \quad (5.31)$$

*Here the remainder term* $\mathrm{o}(\alpha^{d+1})$ *is independent of* $x$ *and the set of points* $\{z_j\}_{j=1}^m$.

**Proof.** From Lemma 5.3 it follows that $E_\alpha - E_0$ satisfies, in $\mathbb{R}^d \setminus \overline{\Omega}$,

$$\begin{cases} \Delta(E_\alpha - E_0) + k_e^2(E_\alpha - E_0) = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ (E_\alpha - E_0) = \alpha^d E_1 + \alpha^{d+1} E_2 + \mathrm{o}\big(\alpha^{d+1}\big) & \text{on } \partial\Omega, \\ \left|\dfrac{\partial}{\partial|x|}(E_\alpha - E_0) - \mathrm{i}k_e(E_\alpha - E_0)\right| = \mathrm{O}\left(\dfrac{1}{|x|^{d-1}}\right). \end{cases}$$

Let $\mathcal{G}$ denote the outgoing Dirichlet Green's function that is defined by:

$$\begin{cases} \Delta\mathcal{G} + k_e^2\mathcal{G} = -\delta & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ \mathcal{G} = 0 & \text{on } \partial\Omega, \\ \left|\dfrac{\partial}{\partial|x|}\mathcal{G} - \mathrm{i}k_e\mathcal{G}\right| = \mathrm{O}\left(\dfrac{1}{|x|^{d-1}}\right). \end{cases}$$

It is easy to see that $E_\alpha - E_0$ has the following integral representation in $\mathbb{R}^d \setminus \overline{\Omega}$:

$$(E_\alpha - E_0)(x) = \int_{\partial\Omega} \frac{\partial\mathcal{G}}{\partial\nu(y)}(x, y)(E_\alpha - E_0)(y)\,\mathrm{d}s(y), \quad \forall x \in \mathbb{R}^d \setminus \overline{\Omega}.$$

Moreover, for any $x \in \mathbb{R}^d \setminus \overline{\Omega}$ which is bounded away from $\partial\Omega$, we obtain from the asymptotic expansion of the boundary condition in Lemma 5.3 that

$$(E_\alpha - E_0)(x) = \alpha^d \int_{\partial\Omega} \frac{\partial\mathcal{G}}{\partial\nu_y}(x, y)(E_1 + \alpha E_2)(y)\,\mathrm{d}s(y) + \mathrm{o}\big(\alpha^{d+1}\big),$$

where $\mathrm{o}(\alpha^{d+1})$ is independent of $x$ and the set of points $\{z_j\}_{j=1}^m$. Since for any $x \in \mathbb{R}^d \setminus \overline{\Omega}$ and $z \in \Omega$ we have by standard integration by parts the following identities:

$$\int_{\partial\Omega} \frac{\partial\mathcal{G}}{\partial\nu}(x, y)\widetilde{G}(y, z)\,\mathrm{d}s(y) = \widetilde{G}(x, z),$$

and

$$\int_{\partial\Omega} \frac{\partial\mathcal{G}}{\partial\nu}(x, y)\,\mathrm{grad}_z\,\widetilde{G}(y, z)\,\mathrm{d}s(y) = \mathrm{grad}_z\,\widetilde{G}(x, z),$$

the expression of the correction term $E_1$ immediately leads to the promised asymptotic expansion. □

Finally, the following asymptotic formula for the scattering amplitude follows from (5.11) and the expansion in Theorem 5.1.

**Theorem 5.2.** *The scattering amplitude has the following asymptotic expansion*:

$$
\begin{aligned}
&A_\alpha\left(\frac{x}{|x|}, q, \omega\right) \\
&\quad = A_0\left(\frac{x}{|x|}, q, \omega\right) + \alpha^d \sum_{j=1}^m \left[\omega^2(\varepsilon_j - \varepsilon_0)\widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right) E_0(z_j)\right. \\
&\qquad\qquad + \left(\frac{1}{\mu_0} - \frac{1}{\mu_j}\right)\left(M_j^{1,1}\left(\frac{\mu_j}{\mu_0}\right)\right. \\
&\qquad\qquad \left.\left.\times \operatorname{grad}_y \widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega)\right) \cdot \operatorname{grad} E_0(z_j)\right]\right. \\
&\qquad + \alpha^{d+1} \sum_{j=1}^m \left[\left(\frac{1}{\mu_0} - \frac{1}{\mu_j}\right)\left(1 - \frac{\mu_0}{\mu_j}\right)\left(\partial^2_{np} E_0(z_j)\big(M_j^{2,1}\big)_{npq}\partial_q \widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right.\right. \\
&\qquad\qquad \left. + \partial_n E_0(z_j)\big(M_j^{1,2}\big)_{npq}\partial^2_{pq}\widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right) \\
&\qquad\qquad + \omega^2(\varepsilon_j - \varepsilon_0)\left(\partial_n E_0(z_j)\left(\int_{B_j} \hat{v}^\mu_{1q}(\xi)\,\mathrm{d}\xi\right)\widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right. \\
&\qquad\qquad \left.\left. + E_0(z_j)\left(\int_{B_j} \xi_n\,\mathrm{d}\xi\right)\partial_n \widetilde{G}_\infty\left(\frac{x}{|x|}, z_j, \omega\right)\right)\right] \\
&\qquad + \mathrm{o}\big(\alpha^{d+1}\big),
\end{aligned}
\tag{5.32}
$$

*for any* $x/|x|$ *and* $q \in S^{d-1} = \{p \in \mathbb{R}^d\colon\ |p|^2 = 1\}$, *where the remainder* $\mathrm{o}(\alpha^{d+1})$ *is independent of* $x/|x|$, $q$, *and the set of points* $\{z_j\}_{j=1}^m$.

## 6. The full Maxwell equations: formal derivations

### 6.1. Asymptotic expansions of the solutions

In this section we give a formal derivation of an asymptotic expansion similar to (2.47)–(2.48) for solutions to the full Maxwell equations. Let $(E_\alpha, H_\alpha)$ denote time-harmonic electromagnetic fields. These time-harmonic fields satisfy the following Maxwell equations:

$$\operatorname{curl} E_\alpha = \mathrm{i}\omega\mu_\alpha H_\alpha \quad \text{in } \Omega,$$

$$\begin{aligned}
&\operatorname{curl} H_\alpha = -\mathrm{i}\omega\varepsilon_\alpha E_\alpha \quad \text{in } \Omega,\\
&E_\alpha \times \nu = f \quad \text{on } \partial\Omega.
\end{aligned} \tag{6.1}$$

Here $f$ is a tangential field on the boundary $\partial\Omega$ that furthermore belongs to the Sobolev space:

$$H_{\mathrm{div}}^{-1/2}(\partial\Omega) = \bigl\{g \in H^{-1/2}(\partial\Omega)^3\colon\ g\cdot\nu = 0 \text{ on } \partial\Omega,\ \operatorname{div}_{\partial\Omega}(g) \in H^{-1/2}(\partial\Omega)\bigr\}.$$

The electromagnetic fields $(E_0, H_0)$ in the absence of any inhomogeneities are solutions to

$$\begin{aligned}
&\operatorname{curl} E_0 = \mathrm{i}\omega\mu_0 H_0 \quad \text{in } \Omega,\\
&\operatorname{curl} H_0 = -\mathrm{i}\omega\varepsilon_0 E_0 \quad \text{in } \Omega,\\
&E_0 \times \nu = f \quad \text{on } \partial\Omega.
\end{aligned} \tag{6.2}$$

For simplicity we only consider one inhomogeneity $z + \alpha B$ and denote the electric permittivity and magnetic permeability inside $z + \alpha B$ by $\varepsilon_*$ and $\mu_*$, respectively. The general case does not lead to any new difficulties. We assume that $\omega$ is not a resonant frequency for the problem (6.2), and so by assumption there exists a unique solution $(E_0, H_0) \in [H(\operatorname{curl}, \Omega)]^2 = \{F \in L^2(\Omega)^3\colon \operatorname{curl} F \in L^2(\Omega)^3\}^2$ of (6.2). It has been shown in [8] that this assumption insures also well-posedness for the $\alpha$-dependent case for $\alpha$ sufficiently small, that is problem (6.1).

To establish a Lippman–Schwinger integral representation formula for the electric field $E_\alpha$ similar to (2.21) we need to introduce the $3 \times 3$ matrix valued function $\mathbf{G}^0$ that is the solution to

$$\begin{aligned}
&\operatorname{curl} \frac{1}{\mu_0} \operatorname{curl} \mathbf{G}^0(x, y) - \omega^2\varepsilon_0 \mathbf{G}^0(x, y) = -\delta_y I \quad \text{in } \Omega,\\
&\mathbf{G}^0(x, y) \times \nu = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{6.3}$$

Here $I$ is the $3 \times 3$ identity matrix.

We note that the above problem has a unique solution since $\omega$ is assumed not to be a resonant frequency for the problem (6.2). For any $x \in \Omega$ we now get, from use of (6.3) and integration by parts,

$$\begin{aligned}
E_\alpha(x) = E_0(x) &+ \mathrm{i}\omega\left(1 - \frac{\mu_*}{\mu_0}\right) \int_{z+\alpha B} \operatorname{curl}_y \mathbf{G}^0(x, y) H_\alpha(y)\, \mathrm{d}y\\
&- \omega^2(\varepsilon_* - \varepsilon_0) \int_{z+\alpha B} \mathbf{G}^0(x, y) E_\alpha(y)\, \mathrm{d}y.
\end{aligned}$$

Exactly as for the TE case, we write the outer expansions as

$$E_\alpha(y) = E_0(y) + \alpha^{\tau_1} E_1(y) + \alpha^{\tau_2} E_2(y) + \cdots,$$
$$H_\alpha(y) = H_0(y) + \alpha^{\tau_1} H_1(y) + \alpha^{\tau_2} H_2(y) + \cdots, \tag{6.4}$$

for $|y - z| \gg \mathrm{O}(\alpha)$, where $0 < \tau_1 < \tau_2 < \cdots$, and $(E_j, H_j)$, $j = 1, 2, \ldots,$ are to be found. Similarly to (2.26) we write the inner expansions as

$$E_\alpha(z + \alpha\xi) = e_\alpha(\xi) = e_0(\xi) + \alpha e_1(\xi) + \alpha^2 e_2(\xi) + \cdots, \quad \text{for } |\xi| = \mathrm{O}(1),$$
$$H_\alpha(z + \alpha\xi) = h_\alpha(\xi) = h_0(\xi) + \alpha h_1(\xi) + \alpha^2 h_2(\xi) + \cdots, \quad \text{for } |\xi| = \mathrm{O}(1), \tag{6.5}$$

where $(e_j, h_j)$, $j = 0, 1, \ldots$ are to be found. The matching conditions are

$$E_0(y) + \alpha^{\tau_1} E_1(y) + \alpha^{\tau_2} E_2(y) + \cdots \sim e_0(\xi) + \alpha e_1(\xi) + \alpha^2 e_2(\xi) + \cdots,$$
$$H_0(y) + \alpha^{\tau_1} H_1(y) + \alpha^{\tau_2} H_2(y) + \cdots \sim h_0(\xi) + \alpha h_1(\xi) + \alpha^2 h_2(\xi) + \cdots. \tag{6.6}$$

If we substitute the inner expansions (6.5) into the Maxwell equations (6.1) and formally equate coefficients of $\alpha^{-1}$ we get from use of the first matching condition in (6.6) that

$$\begin{aligned} &\operatorname{curl} e_0(\xi) = 0 \quad \text{in } \mathbb{R}^3, \\ &\operatorname{div} \varepsilon(\xi) e_0(\xi) = 0 \quad \text{in } \mathbb{R}^3, \\ &e_0(\xi) \to E_0(z) \quad \text{as } |\xi| \to +\infty, \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} &\operatorname{curl} h_0(\xi) = 0 \quad \text{in } \mathbb{R}^3, \\ &\operatorname{div} \mu(\xi) h_0(\xi) = 0 \quad \text{in } \mathbb{R}^3, \\ &h_0(\xi) \to H_0(z) \quad \text{as } |\xi| \to +\infty. \end{aligned} \tag{6.8}$$

Therefore,

$$e_0(\xi) = \big(E_0(z) \cdot \mathbf{u}_i\big) \operatorname{grad} \hat{v}_{1i}^{1/\varepsilon}(\xi), \qquad h_0(\xi) = \big(H_0(z) \cdot \mathbf{u}_i\big) \operatorname{grad} \hat{v}_{1i}^{1/\mu}(\xi).$$

These leading-order terms in the asymptotic expansions of the solutions $E_\alpha$ and $H_\alpha$ to the full Maxwell equations have been rigorously derived by Ammari et al. [8].

The second matching condition in (6.6) yields:

$$\begin{aligned} &\operatorname{curl} e_1(\xi) = \mathrm{i}\omega\mu(\xi) h_0(\xi) \quad \text{in } \mathbb{R}^3, \\ &\operatorname{div}\big(\varepsilon(\xi) e_1(\xi)\big) = 0 \quad \text{in } \mathbb{R}^3, \\ &e_1(\xi) - \operatorname{grad} E_0(z) \cdot \xi \to 0 \quad \text{as } |\xi| \to +\infty, \end{aligned} \tag{6.9}$$

and

$$
\begin{aligned}
&\operatorname{curl} h_1(\xi) = -\mathrm{i}\omega\varepsilon(\xi) e_0(\xi) \quad \text{in } \mathbb{R}^3,\\
&\operatorname{div}\big(\mu(\xi) h_1(\xi)\big) = 0 \quad \text{in } \mathbb{R}^3,\\
&h_1(\xi) - \operatorname{grad} H_0(z)\cdot\xi \to 0 \quad \text{as } |\xi| \to +\infty.
\end{aligned} \tag{6.10}
$$

Note that

$$
\begin{aligned}
&\operatorname{div}\big(\operatorname{grad} E_0(z)\cdot\xi\big) = \operatorname{div}\big(\operatorname{grad} H_0(z)\cdot\xi\big) = 0 \quad \text{in } \mathbb{R}^3,\\
&\operatorname{curl}\big(\operatorname{grad} E_0(z)\cdot\xi\big) = \operatorname{curl} E_0(z) \quad \text{in } \mathbb{R}^3,\\
&\operatorname{curl}\big(\operatorname{grad} H_0(z)\cdot\xi\big) = \operatorname{curl} H_0(z) \quad \text{in } \mathbb{R}^3.
\end{aligned}
$$

Define $\hat{v}_{2i}^{\mu,\varepsilon}$ as the unique solution to

$$
\begin{aligned}
&\hat{v}_{2i}^{\mu,\varepsilon}(\xi) = \mu_0 \operatorname{grad}\big(\hat{v}_{1i}^{1/\mu} - \xi_i\big) \quad \text{in } \mathbb{R}^3 \setminus \overline{B},\\
&\operatorname{curl} \hat{v}_{2i}^{\mu,\varepsilon}(\xi) = \mu_* \operatorname{grad} \hat{v}_{1i}^{1/\mu}(\xi) - \mu_0 \mathbf{u}_i \quad \text{in } B,\\
&\operatorname{div}\big(\varepsilon(\xi)\hat{v}_{2i}^{\mu,\varepsilon}(\xi)\big) = 0 \quad \text{in } \mathbb{R}^3,\\
&\hat{v}_{2i}^{\mu,\varepsilon}(\xi) \to 0 \quad \text{as } |\xi| \to +\infty,
\end{aligned}
$$

and the scalar function $\hat{v}_{2ij}^{\varepsilon}$ to be the unique solution to

$$
\begin{aligned}
&\operatorname{div} \varepsilon \operatorname{grad} \hat{v}_{2ij}^{\varepsilon}(\xi) = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B} \quad \text{and in } B,\\
&\varepsilon_0 \frac{\partial \hat{v}_{2ij}^{\varepsilon}}{\partial \nu}\bigg|_{+}(\xi) - \varepsilon_* \frac{\partial \hat{v}_{2ij}^{\varepsilon}}{\partial \nu}\bigg|_{-}(\xi) = (\varepsilon_0 - \varepsilon_*)\xi_i \nu_j \quad \text{on } \partial B,\\
&\hat{v}_{2ij}^{\varepsilon}(\xi) \to 0 \quad \text{as } |\xi| \to +\infty.
\end{aligned}
$$

From (6.9) and (6.10) we can obtain:

$$
\begin{aligned}
&e_1(\xi) = \mathrm{i}\omega\big(H_0(z)\cdot\mathbf{u}_i\big)\hat{v}_{2i}^{\mu,\varepsilon}(\xi) + \partial_i(E_0\cdot\mathbf{u}_i)(z) \operatorname{grad} \hat{v}_{2ij}^{\varepsilon}(\xi),\\
&h_1(\xi) = -\mathrm{i}\omega\big(E_0(z)\cdot\mathbf{u}_i\big)\hat{v}_{2i}^{\varepsilon,\mu}(\xi) + \partial_i(H_0\cdot\mathbf{u}_i)(z) \operatorname{grad} \hat{v}_{2ij}^{\mu}(\xi),
\end{aligned} \tag{6.11}
$$

where the functions $\hat{v}_{2i}^{\varepsilon,\mu}$ and $\hat{v}_{2ij}^{\mu}$ are defined analogously to $\hat{v}_{2i}^{\mu,\varepsilon}$ and $\hat{v}_{2ij}^{\varepsilon}$ respectively by interchanging the electromagnetic parameters $\varepsilon$ and $\mu$.

### 6.2. *Resonant frequencies*

In this section we provide a formal derivation of an asymptotic formula for the perturbations in the resonant frequencies caused by the presence of a finite number of dielectric inhomogeneities of small volume. The leading-order term in this formula was first derived and rigorously proven in [9].

Let $\omega_0$ be a resonant frequency of multiplicity $n$ of the Maxwell equations in the absence of any inhomogeneities and let $E_0^j$, for $j = 1, \ldots, n$, denote the corresponding eigenfunction, that is the solution to

$$
\begin{aligned}
&\operatorname{curl} \frac{1}{\mu_0} \operatorname{curl} E_0^j - \omega_0^2 \varepsilon_0 E_0^j = 0 \quad \text{in } \Omega, \\
&E_0^j \times \nu = 0 \quad \text{on } \partial\Omega, \\
&\int_\Omega \varepsilon_0 \big(E_0^j\big)^2 = 1.
\end{aligned}
$$

It is known [9] that for $\alpha$ small enough there exist $n$ eigenfrequencies $\{\omega_\alpha^j\}_{j=1}^n$ (counted according to multiplicity) for the Maxwell equations in the presence of small imperfections that converge to $\omega_0$ as $\alpha$ approaches 0.

For $j = 1, \ldots, n$, define $v_\alpha^j$ to be the unique solution of

$$
\begin{aligned}
&\operatorname{curl} \frac{1}{\mu_\alpha} \operatorname{curl} v_\alpha^j = \varepsilon_0 E_0^j \quad \text{in } \Omega, \\
&\operatorname{div}\big(\varepsilon_\alpha v_\alpha^j\big) = 0 \quad \text{on } \partial\Omega, \\
&v_\alpha^j \times \nu = 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

According to [47] we have that

$$
\begin{aligned}
\frac{1}{\omega_0^2} - \frac{1}{n} \sum_{j=1}^n \frac{1}{(\omega_\alpha^j)^2} = {}& \frac{1}{n} \sum_{j=1}^n \frac{1}{\omega_0^2} \left( \frac{1}{\mu_*} - \frac{1}{\mu_0} \right) \int_{z+\alpha B} \operatorname{curl} v_\alpha^j \cdot \operatorname{curl} E_0^j \\
&+ (\varepsilon_0 - \varepsilon_*) \int_{z+\alpha B} v_\alpha^j \cdot E_0^j + \mathrm{O}\big(\alpha^6\big).
\end{aligned}
$$

Set $w_\alpha^j = \frac{1}{\mu_\alpha} \operatorname{curl} v_\alpha^j$. It is readily seen that

$$
\begin{aligned}
&\operatorname{curl} w_\alpha^j = \varepsilon_0 E_0^j \quad \text{in } \Omega, \qquad \operatorname{curl} v_\alpha^j = \mu_\alpha w_\alpha^j \quad \text{in } \Omega, \\
&\operatorname{div}\big(\varepsilon_\alpha v_\alpha^j\big) = 0 \quad \text{in } \Omega, \qquad \operatorname{div}\big(\mu_\alpha w_\alpha^j\big) = 0 \quad \text{in } \Omega, \qquad v_\alpha^j \times \nu = 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\operatorname{curl}\left( w_\alpha^j + \frac{1}{\mathrm{i}\omega_0} H_0^j \right) = 0 \quad \text{in } \Omega, \\
&\operatorname{div}\big(\mu_\alpha w_\alpha^j\big) = 0 \quad \text{in } \Omega, \\
&\left( w_\alpha^j + \frac{1}{\mathrm{i}\omega_0} H_0^j \right) \cdot \nu = 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Here $H_0^j = (\mathrm{i}\omega_0\mu_0)^{-1}\,\mathrm{curl}\,E_0^j$, $j = 1,\ldots,n$. It is also easily seen that

$$\begin{aligned}
&\mathrm{curl}\left(v_\alpha^j - \frac{1}{\omega_0^2}E_0^j\right) = \mu_\alpha w_\alpha^j - \mu_0 w_0^j \quad \text{in } \Omega,\\
&\mathrm{div}\big(\varepsilon_\alpha v_\alpha^j\big) = 0 \quad \text{in } \Omega,\\
&v_\alpha^j \times \nu = 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

Since $\mathrm{div}(w_\alpha^j + (\mathrm{i}\omega_0)^{-1}H_0^j) = 0$ in $\Omega$ we obtain that

$$w_\alpha^j + \frac{1}{\mathrm{i}\omega_0}H_0^j = \mathrm{grad}\,\phi_\alpha^j,$$

where the function $\phi_\alpha^j$ satisfies

$$\begin{aligned}
&\Delta\phi_\alpha^j = 0 \quad \text{in } \Omega \setminus \overline{(z+\alpha B)} \text{ and in } z+\alpha B,\\
&\mu_0 \frac{\partial\phi_\alpha^j}{\partial\nu}\bigg|_+ - \mu_* \frac{\partial\phi_\alpha^j}{\partial\nu}\bigg|_- = \frac{1}{\mathrm{i}\omega_0}(\mu_0 - \mu_*)H_0^j\cdot\nu \quad \text{on } \partial(z+\alpha B),\\
&\frac{\partial\phi_\alpha^j}{\partial\nu} = 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

Clearly we have:

$$\begin{aligned}
&\mathrm{curl}\left(v_\alpha^j - \frac{1}{\omega_0^2}E_0^j\right) = \mu_0\,\mathrm{grad}\,\phi_\alpha^j \quad \text{in } \Omega \setminus \overline{(z+\alpha B)},\\
&\mathrm{curl}\left(v_\alpha^j - \frac{1}{\omega_0^2}E_0^j\right) = \mu_*\,\mathrm{grad}\,\phi_\alpha^j + \frac{1}{\mathrm{i}\omega_0}(\mu_0-\mu_*)H_0^j \quad \text{in } z+\alpha B,\\
&\mathrm{div}\left(\varepsilon_\alpha\left(v_\alpha^j - \frac{1}{\omega_0^2}E_0^j\right)\right) = 0 \quad \text{in } \Omega\backslash\{\overline{z+\alpha B}\} \text{ and } z+\alpha B,\\
&\varepsilon_0\left(v_\alpha^j - \frac{1}{\omega_0^2}E_0^j\right)\bigg|_+ \cdot\nu - \varepsilon_*\left(v_\alpha^j - \frac{1}{\omega_0^2}E_0^j\right)\bigg|_- \cdot\nu = -\frac{1}{\omega_0^2}(\varepsilon_0-\varepsilon_*)E_0^j\cdot\nu \quad \text{on } \partial(z+\alpha B),\\
&\left(v_\alpha^j - \frac{1}{\omega_0^2}E_0^j\right)\times\nu = 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

Rewriting

$$\begin{aligned}
\int_{z+\alpha B} \mathrm{curl}\,v_\alpha^j\cdot\mathrm{curl}\,E_0^j &= \int_{z+\alpha B} \mu_\alpha w_\alpha^j\cdot\mathrm{curl}\,E_0^j = \mathrm{i}\omega_0\mu_0 \int_{z+\alpha B} \mu_\alpha w_\alpha^j\cdot H_0^j\\
&= \mathrm{i}\omega_0\mu_0 \int_{z+\alpha B} \mu_\alpha\,\mathrm{grad}\,\phi_\alpha^j\cdot H_0^j - \mu_0 \int_{z+\alpha B} \mu_\alpha H_0^j\cdot H_0^j
\end{aligned}$$

and deriving formally the following asymptotic expansions

$$\begin{aligned}\phi_\alpha^j(x) = {}&-\alpha\big(H_0^j(z)\cdot\mathbf{u}_l\big)\frac{1}{\mathrm{i}\omega_0}\left(\hat{v}_{1l}^\mu\left(\frac{x-z}{\alpha}\right)-\left(\frac{x-z}{\alpha}\right)_l\right)\\ &+\alpha^2\partial_k\big(H_0^j\cdot\mathbf{u}_l\big)(z)\frac{1}{\mathrm{i}\omega_0}\hat{v}_{2kl}^\mu\left(\frac{x-z}{\alpha}\right)+\mathrm{O}\big(\alpha^3\big),\quad \text{for } x\in\Omega\end{aligned}$$

and

$$\begin{aligned}v_\alpha^j(x) = {}&\frac{1}{\omega_0^2}E_0^j(x)-\alpha\big(H_0^j(z)\cdot\mathbf{u}_l\big)\frac{1}{\mathrm{i}\omega_0}\hat{v}_{2l}^{\mu,\varepsilon}\left(\frac{x-z}{\alpha}\right)\\ &+\frac{\alpha}{\omega_0^2}\big(E_0^j(z)\cdot\mathbf{u}_l\big)\operatorname{grad}_\xi\left(\hat{v}_{1l}^\varepsilon\left(\frac{x-z}{\alpha}\right)-\left(\frac{x-z}{\alpha}\right)_l\right)\\ &-\alpha\omega_0^2\partial_k\big(E_0^j\cdot\mathbf{u}_l\big)(z)\operatorname{grad}_\xi\hat{v}_{2kl}^\varepsilon\left(\frac{x-z}{\alpha}\right)+\mathrm{O}\big(\alpha^2\big),\quad \text{for } x\in\Omega,\end{aligned}$$

to obtain the following accurate asymptotic formula for the resonant frequencies,

$$\begin{aligned}&\frac{1}{\omega_0^2}-\frac{1}{n}\sum_{j=1}^n\frac{1}{(\omega_\alpha^j)^2}\\ &\quad=\frac{1}{n}\sum_{j=1}^n\frac{1}{\omega_0^2}\left(\frac{1}{\mu_*}-\frac{1}{\mu_0}\right)\left[-\alpha^3\big(H_0^j(z)\cdot\mathbf{u}_l\big)\frac{1}{\mathrm{i}\omega_0}\left(\int_B\operatorname{grad}_\xi\hat{v}_{1l}^\mu(\xi)\,\mathrm{d}\xi\right)\cdot\operatorname{curl}E_0^j(z)\right.\\ &\qquad\left.+\alpha^4\partial_k\big(H_0^j\cdot\mathbf{u}_l\big)(z)\frac{1}{\mathrm{i}\omega_0}\left(\int_B\operatorname{grad}_\xi\hat{v}_{2kl}^\mu(\xi)\,\mathrm{d}\xi\right)\cdot\operatorname{curl}E_0^j(z)\right]\\ &\qquad+(\varepsilon_0-\varepsilon_*)\left[\frac{\alpha^3}{\omega_0^2}E_0^j(z)\cdot E_0^j(z)-\alpha^4\big(H_0^j(z)\cdot\mathbf{u}_l\big)\frac{1}{\mathrm{i}\omega_0}\left(\int_B\hat{v}_{2l}^{\mu,\varepsilon}(\xi)\,\mathrm{d}\xi\right)\cdot E_0^j(z)\right.\\ &\qquad+\frac{\alpha^4}{\omega_0^2}\big(E_0^j(z)\cdot\mathbf{u}_l\big)\left(\int_B\operatorname{grad}_\xi\hat{v}_{1l}^\varepsilon(\xi)\,\mathrm{d}\xi\right)\cdot E_0^j(z)\\ &\qquad\left.-\alpha^4\omega_0^2\partial_k\big(E_0^j\cdot\mathbf{u}_l\big)(z)\left(\int_B\operatorname{grad}_\xi\hat{v}_{2kl}^\varepsilon(\xi)\,\mathrm{d}\xi\right)\cdot E_0^j(z)\right]\\ &\qquad+\mathrm{O}\big(\alpha^5\big).\end{aligned}$$

### 6.3. *Scattering amplitude*

In this section we consider the scattering problem in whole of $\mathbb{R}^3$:

$$\begin{aligned}
&\operatorname{curl}\frac{1}{\tilde{\mu}_\alpha}\operatorname{curl}E_\alpha-\omega^2\tilde{\varepsilon}_\alpha E_\alpha=0\quad\text{in }\mathbb{R}^d,\\
&\left|\frac{\partial}{\partial|x|}(E_\alpha-E_{in})-\mathrm{i}k_e(E_\alpha-E_{in})\right|=\mathrm{O}\left(\frac{1}{|x|^{d-1}}\right)\quad\text{as }|x|\to+\infty,
\end{aligned}\tag{6.12}$$

where $E_{in}(x)=\mathrm{i}k_e(q\times p)\times q\mathrm{e}^{\mathrm{i}k_e q\cdot x}$. Here $q\in\mathbb{R}^d$ is a unit vector giving the direction of propagation and $p$ is a constant vector giving the polarization.

Define the scattering amplitude $A_\alpha(x/|x|,p,q,\omega)$ by:

$$E_\alpha(x)=E_{in}(x)+\frac{\mathrm{e}^{\mathrm{i}k_e|x|}}{|x|}A_\alpha\left(\frac{x}{|x|},p,q,\omega\right)+\mathrm{O}\left(\frac{1}{|x|^2}\right)$$

as $|x|\to+\infty$.

We formally derive an asymptotic formula for $A_\alpha$ generalizing (5.14). Using the outgoing Green tensor $\widetilde{\mathbf{G}}$ defined by:

$$\begin{aligned}
&\operatorname{curl}\frac{1}{\tilde{\mu}}\operatorname{curl}\widetilde{\mathbf{G}}(x,y)-\omega^2\tilde{\varepsilon}\widetilde{\mathbf{G}}(x,y)=-\delta_y I\quad\text{in }\mathbb{R}^3,\\
&\left|\frac{\partial}{\partial|x|}\widetilde{\mathbf{G}}(x,y)-\mathrm{i}k_e\widetilde{\mathbf{G}}(x,y)\right|=\mathrm{O}\left(\frac{1}{|x|^2}\right)\quad\text{as }|x|\to+\infty,
\end{aligned}\tag{6.13}$$

we obtain the following Lippman–Schwinger integral representation formula:

$$\begin{aligned}
E_\alpha(x)={}&E_0(x)+\mathrm{i}\omega\left(1-\frac{\mu_*}{\mu_0}\right)\int_{z+\alpha B}\operatorname{curl}_y\widetilde{\mathbf{G}}(x,y)H_\alpha(y)\,\mathrm{d}y\\
&-\omega^2(\varepsilon_*-\varepsilon_0)\int_{z+\alpha B}\widetilde{\mathbf{G}}(x,y)E_\alpha(y)\,\mathrm{d}y,\quad\forall x\in\mathbb{R}^3,
\end{aligned}$$

where $H_\alpha(y)=(\mathrm{i}\omega\tilde{\mu}_\alpha(y))^{-1}\operatorname{curl}E_\alpha(y)$ in $\mathbb{R}^3$.

Based on our results in Section 6.1 of this appendix we can formally establish that

$$\begin{aligned}
E_\alpha(x)={}&E_0(x)+\alpha^3\mathrm{i}\omega\left(1-\frac{\mu_*}{\mu_0}\right)\operatorname{curl}_y\widetilde{\mathbf{G}}(x,z)\cdot\left(\int_B\operatorname{grad}\hat{v}_{1j}^{1/\mu}(\xi)\,\mathrm{d}\xi\right)\big(H_0(z)\cdot\mathbf{u}_j\big)\\
&-\alpha^3\omega^2(\varepsilon_*-\varepsilon_0)\widetilde{\mathbf{G}}(x,z)\left(\int_B\operatorname{grad}v_{1j}^{1/\varepsilon}(\xi)\,\mathrm{d}\xi\right)\big(E_0(z)\cdot\mathbf{u}_j\big)\\
&+\mathrm{O}\big(\alpha^4\big),\quad\forall x\in\mathbb{R}^3\setminus\overline{\Omega},
\end{aligned}$$

to arrive at the following asymptotic expansion of the scattering amplitude $A_\alpha$ in terms of $A_0$:

$$
\begin{aligned}
A_\alpha\left(\frac{x}{|x|}, p, q, \omega\right) = {} & A_0\left(\frac{x}{|x|}, p, q, \omega\right) \\
& + \alpha^3 \mathrm{i}\omega\left(1 - \frac{\mu_*}{\mu_0}\right) \operatorname{curl}_y \widetilde{\mathbf{G}}_\infty\left(\frac{x}{|x|}, z\right) \\
& \times \left(\int_B \operatorname{grad} \hat{v}_{1j}^{1/\mu}(\xi)\,\mathrm{d}\xi\right)\big(H_0(z)\cdot \mathbf{u}_j\big) \\
& - \alpha^3\omega^2(\varepsilon_* - \varepsilon_0)\widetilde{\mathbf{G}}_\infty\left(\frac{x}{|x|}, z\right)\left(\int_B \operatorname{grad} v_{1j}^{1/\varepsilon}(\xi)\,\mathrm{d}\xi\right)\big(E_0(z)\cdot \mathbf{u}_j\big) \\
& + \mathrm{O}\big(\alpha^4\big),
\end{aligned}
$$

where $\widetilde{\mathbf{G}}_\infty(x/|x|, y)$ is defined by:

$$
\widetilde{\mathbf{G}}(x, y) = \frac{\mathrm{e}^{\mathrm{i}k_e|x|}}{|x|}\widetilde{\mathbf{G}}_\infty\left(\frac{x}{|x|}, y\right) + \mathrm{O}\left(\frac{1}{|x|^2}\right)
$$

as $|x| \to +\infty$.


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